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Representations of pointed Hopf algebras and their Drinfel'd quantum doubles

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ABSTRACT

We study representations of nontrivial liftings of nilpotent type of quantum linear spaces and their Drinfel'd quantum doubles. We construct a family of Verma-type modules in both cases and prove a parametrization theorem for simple modules. We compute the Loewy and socle series of Verma modules under a mild restriction on the datum of a lifting. We find bases and dimensions of simple modules.

Introduction

Let H be a finite-dimensional Hopf algebra over a field $\Bbbk$. The goal of this paper is two-fold. First, we want to describe structure of a family of Verma-type H-modules, when H is a certain lifting of a quantum linear space, which entails determination of all simple H-modules. In the second place we carry out a similar program for the Drinfel'd quantum double of H.

We survey some previous related work. There has been significant interest in recent years in representation theory of nonsemisimple Hopf algebras and their quantum doubles [2,33,9,15,18,19,10,29]. In the general setting of such algebra the primary focus is on classifying all simple H-modules in terms of simple modules of the coradical H_0 of H [28,19]. When $H_0 = \Bbbk G$, where $G = G(H)$ is the group of grouplike elements of H, and G is abelian with $\Bbbk$ a splitting field for G of characteristic zero, the simple H_0-modules are given by the elements of the dual group $\widehat{G}$. The problem of estab-

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lishing a bijection between $\widehat{G}$ and the set of isomorphism classes of simple H-modules will be called parametrization of simple H-modules. A typical example of pointed Hopf algebra with abelian group of grouplikes is provided by the fundamental classification result of Andruskiewitsch and Schneider [5]. Recently Radford and Schneider [29] proved the parametrization property for every algebra of the form $u(\mathcal{D}, \lambda)$ from the classification. This result generalizes an earlier theorem of Lusztig involving his small quantum groups [21]. The aforementioned theorems invariably involve a Hopf algebra A with a triangular decomposition

$$A = A^- \otimes A_0 \otimes A^+ \tag{0.1}$$

satisfying the following conditions:

$$\begin{array}{l} A_0 \text{ is a sub-Hopf algebra, } A^- \text{ and } A^+ \text{ are subalgebras} \\ \text{stable under } \mathrm{ad}_\ell\text{- and } \mathrm{ad}_r\text{-action of } A_0, \quad \text{and} \end{array} \tag{0.2}$$

$$A^- = \Bbbk \oplus J^-, \qquad A^+ = \Bbbk \oplus J^+ \tag{0.3}$$

where J^-, J^+ are nilpotent ideals of A^-, A^+, respectively. In order to reach simple A-modules we follow a well-worn path of restriction and induction between A and the subalgebra $A^{\geqslant 0} := A_0 \otimes A^+$. By analogy with the classical Lie theory we call A-modules induced from 1-dimensional $A^{\geqslant 0}$-modules the Verma-type modules.

Let H_i denote the ith term of the coradical filtration of H. By e.g. [18] H_1 is a free H_0-module on a basis $\{1, x_1, \ldots, x_n\}$ where x_i are some skew-primitive elements. When H is generated by H_1 we say that n is the *rank* of H. The simplest family of Hopf algebras within the class of pointed Hopf algebras H with abelian $G(H)$ are liftings of quantum linear spaces completely described in [3]. These Hopf algebras are natural generalizations of Lusztig's small quantum group $u_q(\mathfrak{sl}_2)$ associated to the simple Lie algebra of rank 1. The concept of (Hopf) rank is applicable to them, and in small ranks $n = 1, 2$ the regular representation of H is described in [18,10]. For special cases of rank 2 liftings the entire finite-dimensional representation theory has been obtained in [9,15].

We give an outline of the results. Let H be a lifting of a quantum linear space, and $x_1, \ldots, x_n$ be skew-primitive generators of H. We say that x_i and x_j are *linked* if $x_i x_j - q_{ij} x_j x_i \neq 0$ for a certain root of unity q_{ij}. Let Γ be the simple graph on the vertex set $\{1, 2, \ldots, n\}$ with the edge set made up by pairs (i, j) such that x_i, x_j are linked. We say that Γ is simply linked if every two vertices are connected by at most one edge. A lifting H is called of *nilpotent type* if all x_i are nilpotent.

The first part of the paper is concerned with algebras satisfying conditions (0.1)–(0.3), especially nilpotent simply linked lifting H of a quantum linear space. In the latter case $H_0 = \Bbbk G(H)$. By analogy with the representation theory of $u_q(\mathfrak{sl}_2)$ we can use either sub-Hopf algebra $H^{\geqslant 0} := H_0 H^+$ or $H^{\leqslant 0} := H_0 H^-$ to construct Verma-type modules $Z(\gamma)$ where γ runs over $\widehat{G}$. From the general Theorem 3.3 we have that $Z(\gamma)$ has a unique maximal submodule $R(\gamma)$, the radical of $Z(\gamma)$. Iterating this procedure gives the radical filtration $\{R^m(\gamma)\}$ of $Z(\gamma)$. The first major result of the paper is a description of $R^m(\gamma)$ and the (Loewy) layers of the filtration. We also show that the socle filtration of $Z(\gamma)$ coincides with the radical filtration.

In the second part of the paper we study representations of the Drinfel'd quantum double $D(H)$ of algebras H as in the first part. The doubling procedure yields a new class of Hopf algebras beyond a generalization of quantum groups in [3,4]. For one thing $D(H)$ is not pointed, and for another it does not have decomposition (0.1). Nevertheless, $D(H)$ retains enough good features for developing a sort of Lie theory for Verma-type modules $I(\lambda)$ where λ runs over the characters of $G \times \widehat{G}$. As in part one, but for different reasons, each $I(\lambda)$ has a unique maximal submodule $R(\lambda)$. This enables us to show that the set of simple $D(H)$-modules has the parametrization property. We proceed to describe the radical filtration of $I(\lambda)$ under a mild restriction on the datum for H, which is void whenever the certain structure constants q_i of H have odd orders. The 'odd order' condition is one way to have all weight spaces of $I(\lambda)$ one-dimensional. When this is the case, the lattice $\Lambda(I(\lambda))$ of $D(H)$-submodules is distributive. This is a very strong property which implies that every submodule of $I(\lambda)$ is a unique sum of some local submodules. Thus the lattice $\Lambda(I(\lambda))$ can be recovered from the partially ordered set $\mathcal{J}$ of local submodules. We close with a classification of elements of $\mathcal{J}$.

A more detailed description of material by sections is as follows. In Section 1 we review the construction of liftings of quantum linear spaces and we develop formulas for skew-derivations associated to linked liftings.

In Section 2 we construct the dual basis to the basis of H obtained above. It transpires that the algebra structure of H^* is that of a nilpotent lifting of the quantum linear space with the grouplikes and characters switched around. However, H^* is not pointed. Its coradical is computed in Section 3.3. For related material see [6].

In Section 3 we first establish a general parametrization theorem for simple A-modules where A satisfies (0.1)–(0.3). We then turn to nilpotent type liftings and determine the structure of the radical filtration of induced modules $Z(\gamma), \gamma \in \widehat{G}$, in Theorem 3.7.

We take up the harder case of Drinfel'd double of H in Section 4. Our calculation of multiplication in $D(H)$ is informed by Lemma 4.1 which says that multiplication in the double of a Hopf algebra generated by grouplikes and skew-primitive elements is expressed in terms of automorphisms and skew-derivations associated to skew-primitives. In the case at hand we compute explicitly those skew-derivations in a series of lemmas in Section 4.1. As a first step toward parametrization theorem for simple $D(H)$-modules we find basic subalgebra in the sense of representation theory of algebras. We then construct a family of induced modules $I(\lambda)$ parametrized by $\lambda \in \widehat{\Gamma}$ where $\Gamma = G \times \widehat{G}$. The action of Γ splits up $I(\lambda)$ into a direct sum of weight subspaces. These are made explicit in Lemma 4.14 leading up to the parametrization Theorem 4.15.

The problem of determining the Loewy filtration of $I(\lambda)$ is finer and it is there that we impose a restriction on datum in Definition 4.16. The key step of our analyses consists in showing that generators of $D(H)$ act as raising and lowering operators on the weight basis of $I(\lambda)$. From this we derive the Loewy structure of $I(\lambda)$ in Theorem 4.28. The distributive case is handled in Theorem 4.31.

1. Preliminaries

1.1. Liftings of V

We fix some notation. Below $\Bbbk$ is a field of characteristic 0 containing all roots of 1 and $\Bbbk^\bullet = \Bbbk \setminus \{0\}$. We denote a finite abelian group by G, let $\widehat{G} := \operatorname{Hom}(G, \Bbbk^\bullet)$ denote the dual group, and we let $\Bbbk G$ stand for the group algebra of G over $\Bbbk$. The order of $g \in G$ is denoted by $|g|$. In particular, for a root of unity $q \in \Bbbk^\bullet$, $|q|$ denotes the order of q. We set $\underline{n} = \{1, 2, \dots, n\}$ and $[n] := \{0, 1, \dots, n-1\}$. For a vector space V and a subset X of V we denote the span of X by (X). The unsubscribed '$\otimes$' means '$\otimes_\Bbbk$'. For all $n, m \in \mathbb{Z}$ with $m \geqslant 0$, $\binom{n}{m}_q$ denotes the Gaussian q-binomial coefficient [17], $(n)_q = \binom{n}{1}_q$ and $(n)_q! = (1)_q \cdots (n)_q$.

We review the construction of the underlying algebras of this paper following [3,4]. They belong to the class of Hopf algebras parametrized by some elements of $G \times \widehat{G}$. The starting point of their construction is a left–left Yetter–Drinfel'd finite-dimensional module V over $\Bbbk G$, or a *YD*-module, for short. This means [4] that V is a left $\Bbbk G$-module and a left $\Bbbk G$-comodule with the G-action preserving G-grading. Let us denote by $\omega : V \to \Bbbk G \otimes V$ the comodule structure map and by '.' the G-action. By [4, Section 1.2] V has a basis $\{v_i \mid i \in \underline{n}\}$, where $n = \dim V$, of G- and $\widehat{G}$-eigenvectors, namely there are $a_i \in G$ and $\chi_i \in \widehat{G}$, $i \in \underline{n}$, such that

$$g.v_i = \chi_i(g) v_i, \tag{1.1}$$

$$\omega(v_i) = a_i \otimes v_i, \tag{1.2}$$

for all $g \in G$. We set

$$\overline{\mathcal{D}} = \big(G, (a_i), (\chi_i) \bigm| i \in \underline{n}\big)$$

and call this tuple a linear datum of *rank n* [3] associated with V. We denote by ${}^G_G\mathcal{YD}$ the category of all *YD*-modules over $\Bbbk G$.

The *YD*-module structure on V extends to a *YD*-structure on $V^{\otimes m}$ for every integer $m \geqslant 0$ by using the diagonal action and the codiagonal coaction of $\Bbbk G$ on a tensor product. Explicitly, this means that

$$g.(v_{i_1}\cdots v_{i_m}) = g.v_{i_1}\cdots g.v_{i_m}, \tag{1.3}$$

$$\omega(v_{i_1}\cdots v_{i_m}) = a_{i_1}\cdots a_{i_m}\otimes v_{i_1}\cdots v_{i_m}, \tag{1.4}$$

for all g and $1 \leqslant i_1,\dots,i_m \leqslant n$. Let $F(V)$ be the free associative algebra generated by V. As $F(V) = \bigoplus_{m\geqslant 0} V^{\otimes m}$, $F(V)$ becomes a graded *YD*-module and it follows easily from the formulas (1.3), (1.4) that $F(V)$ is an algebra in ${}^G_G\mathcal{YD}$. Moreover, $F(V)$ is endowed with a special structure of Hopf algebra in ${}^G_G\mathcal{YD}$ [4, Section 2.1]. This is done as follows. First, it is well known [30,4] that the ${}^G_G\mathcal{YD}$ is a braided tensor category with the tensor product just defined and the braiding given by the formula

$$c(u\otimes v) = u_{(-1)}.v\otimes u_{(0)},$$

where we write $\omega(u) = u_{(-1)}\otimes u_{(0)}$ for all $u\in F(V)$. Using this braiding we define the multiplication '$\bullet$' in $F(V)\otimes F(V)$ by

$$(x\otimes y)\bullet(u\otimes v) = x(y_{(-1)}.u)\otimes y_{(0)}v. \tag{1.5}$$

Second, by a straightforward verification (see also [25, §10.5]) the definition (1.5) turns $F(V)\otimes F(V)$ into an algebra in ${}^G_G\mathcal{YD}$ denoted by $F(V)\underline{\otimes} F(V)$. Since $F(V)$ is a free algebra there is an algebra homomorphism

$$\delta : F(V)\to F(V)\underline{\otimes} F(V)$$

defined on the generators by $\delta(v_i) = 1\otimes v_i + v_i\otimes 1$ for $i\in\underline{n}$. Another verification shows that δ is G-linear and G-colinear. All in all we see that $F(V)$ is a bialgebra in ${}^G_G\mathcal{YD}$. Further by [32, 11.0.10] the coalgebra $F(V)$ has the coradical $\Bbbk$, and then an argument of Takeuchi [25, 5.2.10] proves existence of the antipode. Thus $F(V)$ is indeed a Hopf algebra in ${}^G_G\mathcal{YD}$.

Multiplication law (1.5) can be elucidated as follows. The formulas (1.3)–(1.4) allow us to associate with a monomial $\underline{v} = v_{i_1}\cdots v_{i_m}$ the bidegree $(\chi_{\underline{v}}, g_{\underline{v}})$ where $\chi_{\underline{v}} = \chi_{i_1}\cdots\chi_{i_m}$ and $g_{\underline{v}} = a_{i_1}\cdots a_{i_m}$. The set $\{\underline{u}\otimes\underline{v}\}$ forms a basis of $F(V)\otimes F(V)$ in which the definition (1.5) takes on the form

$$(\underline{x}\otimes\underline{y})\bullet(\underline{u}\otimes\underline{v}) = \chi_{\underline{u}}(g_{\underline{y}})\underline{x}\underline{u}\otimes\underline{y}\underline{v}. \tag{1.6}$$

Eq. (1.6) shows that the definition of $F(V)$ is analogous to Lusztig's definition of algebra $'\mathbf{f}$ [20]. Moreover, when G is generated by the a_i's and the mapping $a_i\mapsto\chi_i$ is a homomorphism $G\to\widehat{G}$, $F(V)$ is exactly the Lusztig's type algebra associated to the bilinear form $(,) : G\times G\to\Bbbk$ defined on the generators by $(a_i, a_j) = \chi_i(a_j)$, for all $i, j\in\underline{n}$.

We can now define a fundamental object of the theory. Let $\mathcal{F}(V) = F(V)\otimes\Bbbk G$ be the vector space made into a Hopf algebra by the smash product and smash coproduct constructions. By [26, Theorem 1] $\mathcal{F}(V)$, denoted by $F(V)\#\Bbbk G$, is indeed an ordinary Hopf algebra whose bialgebra structure is described by

$$(u\# g)(v\# h) = u(g.v)\# gh, \tag{1.7}$$

$$\Delta(u\# g) = u^{(1)}\#\left(u^{(2)}\right)_{(-1)}g\otimes\left(u^{(2)}\right)_{(0)}\# g, \tag{1.8}$$

where we write coproduct of $F(V)$ by $\delta(u) = u^{(1)}\otimes u^{(2)}$.

The algebras of interest to us are tied to a special kind of *YD*-module.

Definition 1.1. (See [3, p. 660].) A *YD*-module V with datum $\overline{\mathcal{D}} = ((a_i), (\chi_i) \mid i \in \underline{n})$ is called a *quantum linear space* if

$$\chi_i(a_j)\chi_j(a_i) = 1, \quad \text{for all } i \neq j, \quad \text{and} \quad |q_i| \neq 1, \quad \text{for all } i. \tag{1.9}$$

From now on we assume that V is a quantum linear space. We let $q_{ij} = \chi_j(a_i)$ for $i \neq j$, $q_i = \chi_i(a_i)$ and $m_i = |q_i|$.

A *datum* (or compatible datum [3]) $\mathcal{D}$ for V is a triple

$$\mathcal{D} = \big(\overline{\mathcal{D}}, (\mu_i), (\lambda_{ij})\big),$$

composed of the linear datum $\overline{\mathcal{D}}$ of V and two sets of scalars $(\mu_i)_{i \in \underline{n}}$ and $(\lambda_{i,j})$ with $i \neq j$, $i, j \in \underline{n}$ such that

$$\mu_i = 0 \quad \text{if } a_i^{m_i} = 1 \quad \text{or} \quad \chi_i^{m_i} \neq \epsilon, \tag{1.10}$$

$$\lambda_{ij} = 0 \quad \text{if } a_i a_j = 1 \quad \text{or} \quad \chi_i \chi_j \neq \epsilon, \tag{1.11}$$

$$\lambda_{ji} = -q_{ji}\lambda_{ij}. \tag{1.12}$$

We will identify v_i with $v_i \# 1$. For a datum $\mathcal{D}$ we define the elements p_i and r_{ij} by

$$p_i = v_i^{m_i} - \mu_i\big(a_i^{m_i} - 1\big), \quad \text{for all } i \in \underline{n},$$

$$r_{ij} = v_i v_j - q_{ij} v_j v_i - \lambda_{ij}(a_i a_j - 1), \quad \text{for all } 1 \leqslant i \neq j \leqslant n.$$

We let $I(\mathcal{D})$ be the two-sided ideal of $\mathcal{F}(V)$ generated by p_i, r_{ij} for $i, j \in \underline{n}$ and we set

$$H(\mathcal{D}) = \mathcal{F}(V)/I(\mathcal{D}).$$

We remark that formula (1.8) implies readily that $\Delta(v_i) = v_i \otimes 1 + a_i \otimes v_i$, thus $S(v_i) = -a_i^{-1} v_i$, where S is the antipode of $\mathcal{F}(V)$. A direct verification yields that p_i is $(a_i^{m_i}, 1)$-primitive [3, 5.1] and likewise r_{ij} is $(a_i a_j, 1)$-primitive, thanks to (1.9). In addition, a routine calculation gives $S(p_i) = -a_i^{-m_i} p_i$ [3, 5.1] and also $S(r_{ij}) = -a_i a_j r_{ij}$. Consequently $I(\mathcal{D})$ is a Hopf ideal, hence $H(\mathcal{D})$ is a Hopf algebra associated to $\mathcal{D}$.

As a point of terminology we recall the meaning of lifting of a Hopf algebra [4]. A pointed Hopf algebra H is a *lifting* of a Hopf algebra K if there is a Hopf algebra isomorphism

$$\operatorname{gr} H \simeq K, \tag{1.13}$$

where $\operatorname{gr} H$ is the graded Hopf algebra associated to the coradical filtration of H. Setting the parameters μ_i and λ_{ij} of $\mathcal{D}$ to zero results in a linear datum $\overline{\mathcal{D}}$. The Hopf algebra $H(\overline{\mathcal{D}})$ has a special place in the theory. It is a biproduct of the braided Hopf algebra $R = F(V)/I(\overline{\mathcal{D}})$ and $\Bbbk G$ and by [3, 5.3] $\operatorname{gr} H(\mathcal{D}) \simeq H(\overline{\mathcal{D}})$. As V determines $H(\overline{\mathcal{D}})$ we call $H(\mathcal{D})$ a lifting of V. We note that [3, 5.5] every lifting of $H(\overline{\mathcal{D}})$ has the form $H(\mathcal{D})$ for a datum $\mathcal{D}$ on a quantum linear space V.

1.2. Skew-derivations

We begin by recalling the concept of left and right skew-derivation [16]. Let A be an algebra, a an element of A and ϕ be an algebra endomorphism of A. The assignments

$$b \mapsto ab - \phi(b)a, \quad \text{for all } b \in A,$$

$$b \mapsto ba - a\phi(b), \quad \text{for all } b \in A,$$

define two linear mappings denoted by ${}_\phi[a,b]$ and $[a,b]_\phi$ and called the left and right ϕ-commutators, respectively. They are a left and right ϕ-derivations, respectively, in the sense of having the property

$$ {}_\phi[a,uv] = {}_\phi[a,u]v + \phi(u)_\phi[a,v], \quad \text{for all } u, v \in A, \tag{1.14}$$

$$[a,uv]_\phi = [a,u]_\phi \phi(v) + u[a,v]_\phi, \quad \text{for all } u, v \in A. \tag{1.15}$$

In applications below the endomorphism ϕ is the inner automorphism $\iota_g : h \mapsto ghg^{-1}$, $h \in H$ induced by an invertible element $g \in H$. We shall use a shorter notation ${}_g[a,b]$ and $[a,b]_g$ for a left/right ι_g-commutators.

We shall need a commutation formula for powers of generators. Let A be an algebra, $a, b, x, y \in A$ and $\lambda, q \in \Bbbk^\bullet$, where a, b are invertible. Suppose

$$gxg^{-1} = qx, \qquad gyg^{-1} = q^{-1}y \quad \text{for } g = a, b, \quad \text{and} \quad {}_b[y,x] = \lambda(ab-1).$$

Lemma 1.2. *For every natural $m \geqslant 1$ the following hold*

(1) $${}_{b^m}\left[y^m, x\right] = \lambda(m)_q\left(q^{m-1}ab - 1\right)y^{m-1},$$

(2) $${}_b\left[y, x^m\right] = \lambda(m)_q x^{m-1}\left(q^{m-1}ab - 1\right).$$

Proof. (1) The formula holds for $m = 1$ by definition. We induct on m assuming the formula holds for a given m. We begin by noting that ${}_{b^m}[y^m, x] = [x, y^m]_{a^{-1}}$. Therefore we can apply (1.15) to carry out the induction step. This gives

$$\left[x, y^{m+1}\right]_{a^{-1}} = y^m[x,y]_{a^{-1}} + \left[x, y^m\right]_{a^{-1}}qy$$

(which by the basis of induction and the induction hypothesis)

$$= \lambda\left[y^m(ab-1) + q(m)_q\left(q^{m-1}ab - 1\right)y^m\right].$$

Since $y^m ab = q^{2m}aby^m$ the right-hand side equals

$$\lambda\left(q^{2m}ab - 1 + q(m)_q\left(q^{m-1}ab - 1\right)\right)y^m = \lambda(m+1)_q\left(q^m ab - 1\right)y^m$$

which gives the desired formula.

Part (2) is proven by a similar (and simpler) argument. $\square$

We proceed to the general case. The formula below is a generalization of the Kac's formula [13, (1.3.1)].

Lemma 1.3. *For all integers j and k*

$$ {}_{b^j}\big[y^j, x^k\big] = \sum_{i=1}^{\min(j,k)} x^{k-i} f_i^{j,k} y^{j-i} $$

holds, where $f_i^{j,k} = \lambda^i \binom{j}{i}_q \binom{k}{i}_q (i)_q! q^{(k-i)(j-i)} \prod_{m=1}^{i} (q^{j+k-m-i}ab - 1)$.

Proof. We can assume $\lambda = 1$ by rescaling x via $x' = x/\lambda$ and return back to x by multiplying the formula by λ^k. The assertion holds for every $j \geqslant 1$ and $k = 1$ by the preceding lemma. We induct on k assuming the lemma holds for every $j \geqslant 1$ for a given k. By (1.14) we carry out the induction step as follows

$$ \begin{aligned} {}_{b^j}\big[y^j, x^{k+1}\big] &= {}_{b^j}\big[y^j, x^k\big]x + b^j x^k b^{-j}{}_{b^j}\big[y^j, x\big] \\ &= {}_{b^j}\big[y^j, x^k\big]x + q^{kj} x^k{}_{b^j}\big[y^j, x\big]. \end{aligned} \tag{1.16} $$

By the preceding lemma and the induction hypothesis the right-hand side of (1.16) equals to

$$ \left(\sum_{i=1}^{\min\{j,k\}} x^{k-i} f_i^{j,k} y^{j-i} \right) x + q^{kj} x^k (j)_q \big(q^{j-1}ab - 1\big) y^{j-1}. \tag{1.17} $$

We apply Lemma 1.2 to $y^{j-i}x$ for every $1 \leqslant i < j$

$$ y^{j-i}x = q^{j-i} x y^{j-i} + (j-i)_q \big(q^{j-i-1}ab - 1\big) y^{j-i-1}, $$

and use $(ab)x = q^2 x(ab)$ to rewrite $f_i^{j,k} x = x \tilde{f}_i^{j,k}$ where

$$ \tilde{f}_i^{j,k} = \binom{j}{i}_q \binom{k}{i}_q (i)_q! q^{(k-i)(j-i)} \prod_{m=1}^{i} \big(q^{j+k+2-m-i}ab - 1\big). $$

This allows us to obtain

$$ x^{k-i} f_i^{j,k} y^{j-i} x = q^{j-i} x^{k+1-i} \tilde{f}_i^{j,k} y^{j-i} + (j-i)_q x^{k+1-(i+1)} f_i^{j,k} \big(q^{j-i-1}ab - 1\big) y^{j-i-1}. \tag{1.18} $$

It follows that

$$ {}_{b^j}\big[y^j, x^{k+1}\big] = \sum_r x^{k+1-r} f_r^{j,k+1} y^{j-r}, $$

with $1 \leqslant r \leqslant k+1$, if $k < j$, and $1 \leqslant r \leqslant j$, otherwise, thus showing that $1 \leqslant r \leqslant \min\{j, k+1\}$. Moreover, by (1.18) $f_r^{j,k+1}$ satisfy the recurrence relation

$$ \begin{aligned} f_1^{j,k+1} &= q^{kj} (j)_q \big(q^{j-1}ab - 1\big) + q^{j-1} \tilde{f}_1^{j,k}, \\ f_r^{j,k+1} &= (j-r+1)_q f_{r-1}^{j,k} \big(q^{j-r}ab - 1\big) + q^{j-r} \tilde{f}_r^{j,k}, \quad \text{for all } 2 \leqslant r \leqslant k, \\ f_{k+1}^{j,k} &= (j-k)_q f_k^{j,k} \big(q^{j-k-1}ab - 1\big) \quad \text{if } k < j. \end{aligned} $$

We will show that $f_r^{j,k+1}, 2 \leqslant r \leqslant k$ has the desired form leaving verification of the other cases to the reader. To this end we note that

$$\binom{j}{r}_q (r)_q! = \binom{j}{r-1}_q (r-1)_q!(j-r+1)_q, \qquad \binom{k}{r}_q = \binom{k}{r-1}_q \frac{(k-r+1)_q}{(r)_q},$$

and

$$\prod_{m=1}^{r} \bigl(q^{j+k+2-r-m}ab-1\bigr) = \prod_{m=1}^{r-1} \bigl(q^{j+k+1-r-m}ab-1\bigr)\bigl(q^{j+k+1-r}ab-1\bigr).$$

Therefore

$$f_r^{j,k+1} = \binom{j}{r-1}_q \binom{k}{r-1}_q (r-1)_q \prod_{m=1}^{r-1} \bigl(q^{j+k+1-r-m}ab-1\bigr)\phi,$$

where ϕ can be written in the form $\phi = (j-r+1)_q q^{(k+1-r)(j-r)}\psi$ with

$$\psi = q^{k+1-r}\bigl(q^{j-r}ab-1\bigr) + \frac{(k-r+1)_q}{(r)_q}\bigl(q^{j+k+1-r}ab-1\bigr).$$

It is a straightforward calculation to deduce that $\psi = (q^{k+j+1-2r}+1)\frac{(k+1)_q}{(r)_q}$, which in turn implies that

$$\phi = (j-r+1)_q q^{(k+1-r)(j-r)} \frac{(k+1)_q}{(r)_q}\bigl(q^{k+j+1-2r}ab-1\bigr),$$

and this completes the proof. $\square$

1.3. *Construction of $H(\mathcal{D})$ via iterated Ore extensions*

Ore extensions are a recurring theme of this paper. We refer to [16] for generalities on this subject and we adopt its notation. A construction of an infinite-dimensional algebra closely related to $H(\mathcal{D})$ as an Ore extension of $\Bbbk G$ is given in [7] in rank 1 and 2. Extending the arguments of [7, pp. 747–748] one can show the following. Let $\mathcal{D}$ be a datum of rank n of a quantum linear space. One can construct n pairs of mappings $\{\alpha_i, \delta_i\}$ consisting of automorphism α_i and a left α_i-derivation δ_i of the iterated skew-polynomial extension $\Bbbk G[v_1; \alpha_1, \delta_1] \cdots [v_{i-1}; \alpha_{i-1}, \delta_{i-1}]$ such that $H(\mathcal{D}) = \Bbbk G[v_1; \alpha_1, \delta_1] \cdots [v_n; \alpha_n, \delta_n]$. As an immediate consequence we recover the basis theorem [3, 5.2], namely the set

$$\bigl\{v_1^{i_1} \cdots v_n^{i_n} g \mid g \in G, 0 \leqslant i_j \leqslant m_j - 1, j \in \underline{n}\bigr\} \tag{1.19}$$

is a basis of $H(\mathcal{D})$. This is the *standard* basis of $H(\mathcal{D})$.

2. Algebra structure of H^*

2.1. A basis for H^*

We assume $\mathcal{D}$ fixed and we abbreviate $H(\mathcal{D})$ to H. We will fix some vector notation. For an n-tuple $\underline{i} = (i_1, \ldots, i_n) \in (Z^{\geqslant 0})^n$ and any n noncommuting variables $v_1, \ldots, v_n$ we put $v^{\underline{i}} := v_1^{i_1} \cdots v_n^{i_n}$. We write $\delta_{\underline{i},\underline{j}} = \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}$, $(\underline{i})! = \prod_{k=1}^n (i_k)_{q_k}$, and $\binom{\underline{i}}{\underline{j}} = \prod_{k=1}^n \binom{i_k}{j_k}_{q_k}$. There $\binom{n}{m}_q$ denotes the Gaussian q-binomial coefficient [17]. We let u_k stand for the kth unit vector $(0 \cdots 1 \cdots 0)$ (kth 1). For two vectors $\underline{i}$ and $\underline{j}$ we write $\underline{i} \leqslant \underline{j}$ if $i_k \leqslant j_k$ for all $k \in \underline{n}$.

Every $\gamma \in \widehat{G}$ gives rise to a functional $\widetilde{\gamma} : H \to \Bbbk$ defined in the standard basis by

$$\widetilde{\gamma}\big(v^{\underline{i}} g\big) = \delta_{\underline{0},\underline{i}} \gamma(g). \tag{2.1}$$

The mapping $\gamma \to \widetilde{\gamma}$ is a group embedding $\widehat{G} \to H^*$, but not a coalgebra map, if the set $\widehat{G}$ is given the grouplike coalgebra structure. Below we identify γ with $\widetilde{\gamma}$ via that embedding.

For every $g \in G$ we associate a minimal idempotent

$$\epsilon_g = \frac{1}{|G|} \sum_{\gamma \in \widehat{G}} \gamma\big(g^{-1}\big)\gamma$$

of $\Bbbk\widehat{G}$. The natural pairing

$$G \times \widehat{G} \to \Bbbk^{\bullet}, \qquad \langle g, \gamma \rangle \mapsto \gamma(g)$$

induces the canonical isomorphism $G \simeq \widehat{\widehat{G}}$. It follows that the set $\{\epsilon_g \mid g \in G\}$ forms a basis of $\Bbbk\widehat{G}$ dual to the standard basis $\{g \mid g \in G\}$ of $\Bbbk G$.

We will find useful to have a formula for straightening out certain products. For $\underline{m} \leqslant \underline{i}$ we define the scalars

$$\phi(\underline{m}, \underline{i}) = \prod_{p=2}^{n} \chi_p^{m_p}\big(a_1^{i_1 - m_1} \cdots a_{p-1}^{i_{p-1} - m_{p-1}}\big).$$

Lemma 2.1. *In the foregoing notation, for every $g \in G$*

$$v_1^{m_1} a_1^{i_1 - m_1} \cdots v_n^{m_n} a_n^{i_n - m_n} g = \phi(\underline{m}, \underline{i}) v^{\underline{m}} a^{\underline{i} - \underline{m}} g. \tag{2.2}$$

Proof. The formula follows immediately from relation (1.7). $\square$

We define the functionals ξ_i, $i \in \underline{n}$, in the standard basis of H by the rule

$$\xi_k\big(v^{\underline{i}} g\big) = \delta_{u_k,\underline{i}} \quad \text{for every } g \in G. \tag{2.3}$$

Lemma 2.2. *For every $c < m_k$*

(i) $\xi_k^c(v^{\underline{i}} g) = (c)_{q_k}! \delta_{c u_k, \underline{i}}$,
(ii) $\xi_k^{m_k} = 0$ *for all $k \in \underline{n}$.*

Proof. We begin by noting that in view of $(a_k \otimes v_k)(v_k \otimes 1) = q_k(v_k \otimes 1)(a_k \otimes v_k)$ the quantum binomial formula [17] gives

$$\Delta\bigl(v_k^{i_k}\bigr) = \sum_{m_k=0}^{i_k} \binom{i_k}{m_k}_{q_k} v_k^{m_k} a_k^{i_k-m_k} \otimes v_k^{i_k-m_k}.$$

It follows from this together with Lemma 2.1 that

$$\Delta\bigl(v^{\underline{i}} g\bigr) = \sum_{\underline{m},\underline{l}} \binom{\underline{i}}{\underline{m}} \phi(\underline{m}, \underline{i}) v^{\underline{m}} a^{\underline{i}-\underline{m}} g \otimes v^{\underline{i}-\underline{m}} g. \tag{2.4}$$

Now (i) holds for $c = 1$ by the definition of ξ_k. Assuming it holds for c, the induction step is as follows:

$$\begin{aligned} \xi_k^{c+1}\bigl(v^{\underline{i}} g\bigr) &= \bigl\langle \xi_k^c \otimes \xi_k, \Delta\bigl(v^{\underline{i}} g\bigr)\bigr\rangle \\ &= \sum \binom{\underline{i}}{\underline{m}} \phi(\underline{m}, \underline{i}) \xi_k^c\bigl(v^{\underline{m}} a^{\underline{i}-\underline{m}} g\bigr) \xi_k\bigl(v^{\underline{i}-\underline{m}} g\bigr). \end{aligned} \tag{2.5}$$

By the induction hypothesis and the basis of induction $\xi_k^c(v^{\underline{m}} a^{\underline{i}-\underline{m}} g) = (c)_{q_k}! \delta_{cu_k, \underline{m}}$, and $\xi_k(v^{\underline{i}-\underline{m}} g) = \delta_{u_k, \underline{i}-\underline{m}}$. It follows readily that the nonzero terms in the right side of (2.5) satisfy $\underline{m} = cu_k$, $\underline{i} - \underline{m} = u_k$. Thus $\underline{i} = (c+1)u_k$. Therefore the sum in (2.5) equals $\binom{(c+1)u_k}{cu_k} \phi(cu_k, (c+1)u_k)$. It remains to note that $\binom{(c+1)u_k}{cu_k} = (c+1)_{q_k}!$ and $\phi(cu_k, (c+1)u_k) = \chi_k^c(a_1^0 \cdots a_{k-1}^0) = 1$.
(ii) follows from (i) as $(m_k)_{q_k} = 0$. $\square$

We can give a formula for the dual basis to the standard basis of H. For related results see [6].

Proposition 2.3. *For every $\underline{c}$ there holds*

(1) $\xi^{\underline{c}}(v^{\underline{i}} g) = (\underline{c})! \delta_{\underline{c}, \underline{i}}$.
(2) *The set*

$$\bigl\{\bigl[(\underline{c})!\bigr]^{-1} \xi^{\underline{c}} \epsilon_g \bigm| 0 \leqslant c_k < m_k \textit{ for all } k \in \underline{n} \textit{ and } g \in G\bigr\}$$

is the dual basis to the standard basis of H.

Proof. We begin with (1). We induct on n, referring to the preceding lemma for the case $n = 1$. Assuming the formula holds for all $\underline{c}$ with $c_n = 0$, take $\underline{c}$ with $c_n \neq 0$, and set $\underline{c}' = (c_1, \ldots, c_{n-1}, 0)$. As $\xi^{\underline{c}} = \xi^{\underline{c}'} \xi_n^{c_n}$ formula (2.4) gives

$$\xi^{\underline{c}}\bigl(v^{\underline{i}} g\bigr) = \sum \binom{\underline{i}}{\underline{m}} \phi(\underline{m}, \underline{i}) \xi^{\underline{c}'}\bigl(v^{\underline{m}} a^{\underline{i}-\underline{m}} g\bigr) \xi_n^{c_n}\bigl(v^{\underline{i}-\underline{m}} g\bigr).$$

By the induction hypothesis and Lemma 2.2 the sum equals

$$(\underline{c}')!(c_n)_{q_n} \binom{\underline{i}}{\underline{m}} \phi(\underline{m}, \underline{i})$$

where $\underline{m} = \underline{c}'$ and $\underline{i} - \underline{m} = c_n u_n$. Thus $\underline{i} = \underline{c}' + c_n u_n = \underline{c}$ and it is easy to check that $\binom{\underline{i}}{\underline{m}} = 1 = \phi(\underline{m}, \underline{i})$. This completes the proof of (1).

We move to (2). Using formula (2.4) we compute

$$\xi^{\underline{c}}\epsilon_h\big(\upsilon^{\underline{i}}g\big) = \sum \binom{\underline{i}}{\underline{m}}\phi(\underline{m},\underline{i})\xi^{\underline{c}}\big(\upsilon^{\underline{m}}a^{\underline{i}-\underline{m}}g\big)\epsilon_h\big(\upsilon^{\underline{i}-\underline{m}}g\big).$$

By part (1) and definition (2.1) the value υ of the above sum is $\upsilon = (\underline{c})!\epsilon_h(g)$, provided $\underline{i} = \underline{m} = \underline{c}$, and zero, otherwise, as needed. □

Proposition 2.4.

(1) *For every $\gamma \in \widehat{G}$ and $1 \leqslant k \leqslant n$*

$$\gamma\xi_k = \gamma(a_k)\xi_k\gamma.$$

(2) *For all $s, t \in \underline{n}$ there holds*

$$\xi_s\xi_t = \chi_s(a_t)\xi_t\xi_s.$$

Proof. To show (1) we compare the values of $\gamma\xi_k(v^{\underline{i}}g)$ and $\xi_k\gamma(v^{\underline{i}}g)$. Using (2.4), the definition of ξ_k and (2.1), the first scalar equals $\gamma(a_kg)$ while the second is $\gamma(g)$, provided $\underline{i} = u_k$, and zero, otherwise.

We proceed to a proof of (2). For $s < t$ the preceding proposition gives $\xi_s\xi_t(v^{\underline{i}}g) = \delta_{u_s+u_t,\underline{i}}$. In the opposite order using (2.4) one can compute easily that $\xi_t\xi_s(v^{\underline{i}}g) = \delta_{u_s+u_t,\underline{i}}\binom{u_s+u_t}{u_t}\phi(u_t, u_s+u_t)$. Noting that $\phi(u_t, u_s+u_t) = \chi_t(a_s)$ by the definition of ϕ we conclude that $\xi_t\xi_s = \chi_t(a_s)\xi_s\xi_t$. But then (2) holds for $\xi_s\xi_t$ as well, because $\chi_t^{-1}(a_s) = \chi_s(a_t)$ by (1.9). □

The last two propositions yield a short alternative proof to [3, 5.3].

Corollary 2.5. *(See [3, 5.3].) Let H be a lifting of a quantum linear space and H_r be the rth term of the coradical filtration of H. There holds*

$$H_r = \left(v_1^{i_1} \cdots v_n^{i_n} \;\middle|\; \sum i_j \leqslant r \text{ and } g \in G \right).$$

Proof. The ideal J of H^* generated by $\xi_i, i \in \underline{n}$, is nilpotent, thanks to Propositions 2.3(2)–2.4. J is the radical of H^* as $H^*/J \simeq \Bbbk\widehat{G}$ by Proposition 2.3. The assertion follows from [25, 5.2.9]. □

3. Triangulated algebras

3.1. A general theorem

In this section we prove a general form of parametrization property for algebras with certain triangular decomposition. Our proof is similar to [19, Thm. 1].

We begin with several preliminary remarks. Let A be a Hopf algebra satisfying the conditions set forth in (0.1)–(0.3). Let us call a subalgebra of A, A_0-normal if it is stable under both adjoint actions of A_0. The restriction of counit ϵ to A^+ has kernel J^+. Since ϵ is A_0-linear with respect to both adjoint actions, J^+ is a normal subalgebra. The identities $ax = \sum(\mathrm{ad}_\ell a_1)(x)a_2$ and $xa = \sum a_1(\mathrm{ad}_r a_2)(x)$ with $a \in A_0, x \in A^+$ show that $A_0A^+ = A^+A_0$, hence $A^{\geqslant 0} := A_0A^+$ is a subalgebra of A. The splitting $A^+ = \Bbbk \oplus J^+$ implies that $A^{\geqslant 0} = A_0 \oplus A_0J^+$. Since J^+ is an A_0-normal subalgebra, $A_0J^+ = J^+A_0$ and therefore A_0J^+ is a nilpotent ideal of $A^{\geqslant 0}$. It follows that all simple $A^{\geqslant 0}$-modules are pullbacks of simple A_0-modules along $A^{\geqslant 0} \to A_0$. For every simple (left) A_0-module V we define A-module $Z(V)$ by the formula

$$Z(V) = A \otimes_{A^{\geqslant 0}} V. \tag{3.1}$$

For an A-module M we denote by M_0 the socle of its restriction to $A^{\geqslant 0}$. We need two auxiliary observations.

Lemma 3.1. *For every A_0-module there holds*

$$Z(V) = V \otimes J^- V.$$

Proof. Condition (0.1) implies readily the decomposition

$$A = A^{\geqslant 0} \oplus J^- A^{\geqslant 0}.$$

Tensoring this direct sum by V over $A^{\geqslant 0}$ gives the desired formula. $\square$

Lemma 3.2. *For every simple A_0-module V the induced module $Z(V)$ has a unique maximal A-submodule contained in $J^- V$.*

Proof. By the preceding lemma and since $A_0 J^- = J^- A_0$ the subspace $J^- V$ is a maximal A_0-submodule of $Z(V)$. Suppose M is a proper A-submodule of $Z(V)$ not contained in $J^- V$. Then $T + J^- V = Z(V)$ and since J^- is nilpotent, the argument of the Nakayama's lemma gives $M = Z(V)$, a contradiction. Now set R equal to the sum of all proper A-submodules of $Z(V)$. $\square$

We denote the maximal submodule of the above lemma by $R(V)$. We define a family of simple A-modules by

$$L(V) = Z(V)/R(V).$$

Theorem 3.3. *The mapping $V \mapsto L(V)$ sets up a bijection between the isomorphism classes of simple A_0-modules and the isomorphism classes of simple A-modules.*

Proof. Let M be a simple A-module. Select a simple left $A^{\geqslant 0}$-submodule V of M. We observe that an $A^{\geqslant 0}$-map $\iota_V : V \to Z(V)$, $\iota_V(v) = 1 \otimes v$ is universal among all $A^{\geqslant 0}$-maps of V in A-modules. Namely, every $f : V \to M$, M is an A-module, can be uniquely extended to $f_* : Z(V) \to M$ satisfying the equality $f_* \iota_V = f$ via $f_*(a \otimes v) = af(v)$. It follows that $M \simeq L(V)$ for some simple A_0-module V. It remains to show that $L(V) \simeq L(U)$ for two simple A_0-modules V and U if and only if $V \simeq U$. To this end it suffices to show that $L(V)_0 = V$.

Let $\nu : Z(V) \to L(V)$ be the natural epimorphism. Set $\overline{V} = \nu(V)$ and notice that since $\operatorname{Ker} \nu = R(V) \subset J^- V$ we have an isomorphism of $A^{\geqslant 0}$-modules $\overline{V} \simeq V$ as well as the decomposition $L(V) = \overline{V} \oplus J^- \overline{V}$. Let π be the A_0-projection of $L(V)$ on $J^- \overline{V}$. Suppose there is a simple $A^{\geqslant 0}$-submodule U of $L(V)$ distinct from $\overline{V}$. Set $U' = \pi(U)$ and notice that U' is a simple A_0-submodule of $J^- \overline{V}$. Evidently $U' \subset U + \overline{V}$, hence $J^+ U' = 0$. Therefore by simplicity of $L(V)$ we have $L(V) = AU' = U' + J^- U'$. It follows that $L(V) = J^- \overline{V}$ hence $L(V) = J^- L(V)$ forcing $L(V) = 0$, a contradiction. $\square$

3.2. Representations of H

Let $\mathcal{D} = (G, (a_i), (\chi_i), (\mu_i), (\lambda_{ij}), i, j \in \underline{N})$ be a datum on a quantum linear N-dimensional space. Following [29, Section 4.1] we associate to $\mathcal{D}$ its *linking graph* $\Gamma(\mathcal{D})$ which is a simple graph with the vertex set $\underline{N}$ and the edge set of all (i, j) such that $\lambda_{ij} \neq 0$. As usual in graph theory the degree of a vertex i is the number $d(i)$ of all j such that $\lambda_{ij} \neq 0$. We say that $\mathcal{D}$ is *simply linked* datum if $d(i) \leqslant 1$ for all i. The simplicity condition is not very severe. For by remark [3, Section 5] $d(i) \leqslant 1$ whenever $|q_i| \geqslant 3$. The vertices of degree zero give rise to generators of $H(\mathcal{D})$ lying in the radical of $H(\mathcal{D})$. For our purposes we can assume that $\Gamma(\mathcal{D})$ does not have such vertices. We call $\mathcal{D}$ and $H(\mathcal{D})$

of *nilpotent type* if $\mu_i = 0$ for every vertex i. From now on $\mathcal{D}$ is a simply linked datum of nilpotent type with every vertex of degree 1. Clearly, the number of vertices N is even, so we set $n = N/2$. Renumbering vertices, if necessary we can assume that the edge set of $\Gamma(\mathcal{D})$ is $\{(i, i+n) \mid i \in \underline{n}\}$. It will be convenient to modify notation. We put $b_i = a_{i+n}$, $x_i = v_i$ and $y_i = v_{i+n}$ for every $i \in \underline{n}$. Rescaling x_i we will assume $\lambda_{i,i+n} = 1$. Thus $\mathcal{D}$ has the form

$$\mathcal{D} = \bigl\{G, (a_i), (b_i), (\chi_i), \lambda_{ij}, \underline{n} \bigm| i, j \in \underline{n}\bigr\}.$$

Now (1.11) implies $a_i b_i \neq 1$ and $\chi_{i+n} = \chi_i^{-1}$ for all $i \in \underline{n}$ which in turn gives the following conditions:

$$\text{(D0)} \qquad a_i b_i \neq 1 \quad \text{for all } i,$$
$$\text{(D1)} \qquad \chi_j(a_i) = \chi_i(b_j) \quad \text{for all } i, j,$$
$$\text{(D2)} \qquad \chi_i(a_j)\chi_j(a_i) = 1 \quad \text{for all } i \neq j,$$
$$\text{(D3)} \qquad \chi_i(b_j)\chi_j(b_i) = 1 \quad \text{for all } i \neq j.$$

The Hopf algebra $H = H(\mathcal{D})$ attached to $\mathcal{D}$ is explicitly described as follows. H is generated by G and $2n$ symbols $\{x_i, y_i, i \in \underline{n}\}$ subject to the relations of G and the following relations:

(R1) $gx_i = \chi_i(g)x_i g$ for all $g \in G$,
(R2) $gy_i = \chi_i^{-1}(g)y_i g$ for all $g \in G$,
(R3) $x_i x_j - q_{ij}x_j x_i = 0$ for $i \neq j$,
(R4) $y_i y_j - q_{ij}y_j y_i = 0$ for $i \neq j$,
(R5) $x_i y_j - q_{ij}^{-1} y_j x_i = \delta_{ij}(a_i b_i - 1)$ for all i,
(R6) $x_i^{m_i} = 0 = y_i^{m_i}$ for all i,
(R7) $\Delta x_i = a_i \otimes x_i + x_i \otimes 1$ for all i,
(R8) $\Delta y_i = b_i \otimes y_i + y_i \otimes 1$ for all i.

We proceed now to a classification of simple H-modules. Our approach is a generalization of [10]. Let Y be the sub-Hopf algebra of H generated by G and $y_i, i \in \underline{n}$. For every $\gamma \in \widehat{G}$ we make $\Bbbk$ a Y-module denoted by $\Bbbk_\gamma$ by setting

$$g.1_\gamma = \gamma(g) \quad \text{for all } g \in G,$$
$$y_i.1_\gamma = 0 \quad \text{for all } i$$

where 1_γ is identified with $1 \in \Bbbk$. We define the H-module $Z(\gamma)$ by inducing from Y to H via

$$Z(\gamma) = H \otimes_Y \Bbbk_\gamma.$$

Since H is a free Y-module on a basis $\{x^{\underline{i}}\}$ we see that the set $\{x^{\underline{i}} \otimes 1_\gamma\}$ forms a basis for $Z(\gamma)$.

Let M be an H-module. We say that $0 \neq m \in M$ is a weight element of weight $\gamma \in \widehat{G}$ if $g.m = \gamma(g)m$ for all $g \in G$. A weight element is called primitive if $y_i.m = 0$ for all i. For $\gamma \in \widehat{G}$ we define $S(\gamma)$ to be the subset of all $j \in \underline{n}$ such that

$$\gamma(a_j b_j) = q_j^{-e_j} \quad \text{for some } 0 \leqslant e_j \leqslant m_j - 2. \tag{3.2}$$

We denote by $e_j(\gamma)$ the above integer, dropping γ whenever it is clear from the context. We say that elements x, y of an algebra skew commute if $xy = qyx$ for some nonzero $q \in \Bbbk$.

Lemma 3.4. *A monomial $x^{\underline{i}} \otimes 1_\gamma$ is primitive if and only if $i_j = 0, e_j + 1$ for every $j \in S(\gamma)$, and $i_k = 0$ for all $k \notin S(\gamma)$.*

Proof. Since y_j skew commutes with every $x_i, i \neq j$, there is $c \in \Bbbk^\bullet$ such that

$$y_j.x^{\underline{i}} \otimes 1_\gamma = c x_1^{i_1} \cdots x_{j-1}^{i_{j-1}} y_j x_j^{i_j} \cdots x_n^{i_n} \otimes 1_\gamma.$$

By Lemma 1.2 $y_j.x_j^{i_j} \otimes 1_\gamma = -q_j (i_j)_{q_j} x_j^{i_j-1} (q_j^{i_j-1} a_j b_j - 1)$, provided $i_j \neq 0$, and 0, otherwise. Further, for every $k \neq j$ the condition (D1) implies that

$$\chi_k(a_j b_j) = \chi_k(a_j)\chi_k(b_j) = \chi_k(a_j)\chi_j(a_k) = 1. \tag{3.3}$$

Therefore $a_j b_j$ commutes with every $x_k, k \neq j$. It follows that $y_j.x^{\underline{i}} \otimes 1_\gamma = 0$ if and only if $i_j = 0, e_j + 1$, the last possibility occurring for $j \in S(\gamma)$ only. □

By Lemma 3.2 each $Z(\gamma)$ has a unique maximal submodule $R(\gamma)$, possibly zero. We associate a simple H-module

$$L(\gamma) = Z(\gamma)/R(\gamma)$$

to every $\gamma \in \widehat{G}$. The next result explicitly describes $R(\gamma)$.

Proposition 3.5. *In the foregoing notation*

(1) *The family $\{L(\gamma) \mid \gamma \in \widehat{G}\}$ is a full set of representatives of simple H-modules.*
(2) *$R(\gamma)$ is the sum of all submodules generated by $x_j^{e_j+1} \otimes 1_\gamma$, $j \in S(\gamma)$.*

Proof. (1) is a particular case of [29, 4.12] or of Theorem 3.3.

(2) On the one hand each primitive vector $x^{\underline{i}} \otimes 1_\gamma$ generates a submodule spanned by all $x^{\underline{j}} \otimes 1_\gamma$ with $\underline{j} \geqslant \underline{i}$, hence a proper one.

Conversely, suppose $v = \sum_{\underline{i}} c_{\underline{i}} x^{\underline{i}} \otimes 1_\gamma, c_{\underline{i}} \in \Bbbk^\bullet$, generates a proper submodule. If $i_j \geqslant e_j + 1$ for at least one $j \in S(\gamma)$, then $x^{\underline{i}} \otimes 1_\gamma$ lies in $H.(x^{e_j+1} \otimes 1_\gamma)$, hence is contained by $R(\gamma)$ by the opening remark. Suppose v involves a monomial $x^{\underline{i}} \otimes 1_\gamma$ with $i_j \leqslant e_j$ for all $j \in S(\gamma)$. By Lemma 1.3

$$y_j^{i_j} x_j^{i_j} \otimes 1_\gamma = -q_j (i_j)_{q_j}! \prod_{m=1}^{i_j} \bigl(q_j^{i_j-m} \gamma(a_j b_j) - 1\bigr) \otimes 1_\gamma.$$

It follows that

$$y_1^{i_1} \cdots y_n^{i_n} v = c_{\underline{i}} d \otimes 1_\gamma + \sum_{\underline{m} \neq \underline{0}} \kappa_{\underline{m}} x^{\underline{m}} \otimes 1_\gamma,$$

with $d \neq 0, \kappa_{\underline{m}} \in \Bbbk$. But the latter element generates $Z(\gamma)$ since x_i is nilpotent for all i. This completes the proof. □

We derive from the previous proposition the dimension of $L(\gamma)$.

Corollary 3.6. $\dim L(\gamma) = \prod_{k \notin S(\gamma)} m_k \prod_{j \in S(\gamma)} (e_j + 1)$.

We proceed to calculation of the radical and socle series of $Z(\gamma)$. First we recall [22] that for a finite-dimensional algebra A and a left A-module M the radical $R(M)$ of M is the smallest submodule such that $M/R(M)$ is semisimple. It is easy to see that $R(M) = JM$ where J is the radical of A. Dually the socle $\Sigma(M)$ of M is the largest semisimple submodule of M. The radical or Loewy series $\{R^n(M)\}$ of M is defined recursively by $R^0(M) = M$ and $R^m(M) = R(R^{m-1}(M))$ for $m \geqslant 1$. Similarly, the socle series $\{\Sigma_m(M)\}$ is defined by $\Sigma_0(M) = 0$ and $\Sigma_m(M)$ is the preimage in M of $\Sigma(M/\Sigma_{m-1}(M))$ for $m \geqslant 1$. We note that the numbers $\min\{m \mid R^m(M) = 0\}$ and $\min\{m \mid \Sigma_m(M) = M\}$ coincide. The common value is known as the Loewy length of M, denoted by $\ell(M)$. Moreover, the two series are related by inclusion $R^m(M) \subseteq \Sigma_{\ell(M)-m}(M)$ for all m. We say that two filtrations of an A-module coincide if every term of one filtration is a term of another.

For $A = H$ and $M = Z(\gamma)$ we write $\ell(\gamma)$ and $R^m(\gamma)$, $\Sigma_m(\gamma)$ for the Loewy length and the terms of the Loewy and the socle series, respectively. We define *the rank* of monomial $x^{\underline{i}} \otimes 1_\gamma$ as the number $\mathrm{rk}(x^{\underline{i}} \otimes 1_\gamma)$ of all $j \in S(\gamma)$ such that $i_j \geqslant e_j(\gamma) + 1$.

Theorem 3.7. (1) *For every $m < \ell(\gamma)$ $R^m(\gamma)$ is generated by the primitive vectors of rank m.*

(2) *The Loewy and the socle filtration coincide.*

(3) *The Loewy layers $\mathbb{L}^m(\gamma) := R^m(\gamma)/R^{m+1}(\gamma)$ are given by*

$$\mathbb{L}^m \simeq \bigoplus \bigl\{ L(\eta) \bigm| \eta \text{ is weight of primitive basis vector of rank } m \bigr\}.$$

(4) $\ell(\gamma) = |S(\gamma)| + 1$.

Proof. (1)–(4). We often drop γ when it is clear from the context. We induct on m. The assertion holds for $m = 0$ as $Z(\gamma)$ is generated by $1 \otimes 1_\gamma$. Suppose it is true for R^m. Pick a primitive vector $v_{\underline{i}} := x^{\underline{i}} \otimes 1_\gamma$. Clearly $\eta = \gamma \chi^{\underline{i}}$ is weight of $v_{\underline{i}}$. By the universal property of induced modules there is an epimorphism $\phi : Z(\eta) \to H.v_{\underline{i}}$ sending $1 \otimes 1_\eta \mapsto v_{\underline{i}}$. It follows that $R(H.v_{\underline{i}}) = \phi(R(\eta))$ which by the previous proposition equals $\sum H\phi(w)$, where w runs over all primitive monomials of $Z(\eta)$ of rank 1. We next compute $S(\eta)$. We have $\eta(a_k b_k) = \gamma \chi_k^{i_k}(a_k b_k)$, because $\chi_l(a_k b_k) = 1$ for $l \neq k$, as in the proof of Lemma 3.4. Noticing that $i_k = 0$ for $k \notin S(\gamma)$ and $i_k = 0$, $e_k(\gamma) + 1$, otherwise, we arrive at $\eta(a_k b_k) = \gamma(a_k b_k)$ if $k \notin S(\gamma)$ and $\eta(a_k b_k) = q_k^{-e_k(\gamma)}$, $q_k^{-(m_k - e_k(\gamma) - 2)}$, otherwise. It follows that $S(\eta) = S(\gamma)$ with $e_k(\eta) = e_k(\gamma)$ or $e_k(\eta) = m_k - e_k(\gamma) - 2$ for all $k \in S(\gamma)$. Furthermore, as $w = x_j^{e_j(\eta)+1} \otimes 1_\eta$ for some $j \in S(\gamma)$ we get $\phi(w) = x_j^{e_j(\eta)+1} v_{\underline{i}}$. Therefore if $i_j \neq 0$, then in view of $m_j = e_j(\eta) + e_j(\gamma) + 2$ and $x_j^{m_j} = 0$, we have $\phi(w) = 0$. Otherwise, $e_j(\eta) = e_j(\gamma)$, hence $\phi(w) = x_j^{e_j(\gamma)+1} v_{\underline{i}}$. It follows that every nonzero $\phi(w)$ has rank $m + 1$. Moreover, every primitive monomial of rank $m + 1$ has the form $\phi(w)$ for a choice of $v_{\underline{i}}$ and j. Noting that $R^{m+1}(\gamma) = \sum R(H.v_{\underline{i}})$ where $v_{\underline{i}}$ runs over all primitive monomials, the induction step is complete. This proves (1) from which (4) is an obvious consequence.

(2) Since $R^\ell = \Sigma_0$ we may assume by the reverse induction on m that part (2) holds for all $k > m$. Set $M = Z(\gamma)/R^{m+1}(\gamma)$. By the induction hypotheses $R^{m+1} = \Sigma_{\ell-m-1}$, hence $\Sigma(M) = \Sigma_{\ell-m}/R^{m+1}$. Therefore, were R^m a proper submodule of $\Sigma_{\ell-m}$ there would be a simple H-module L of M not contained in R^m/R^{m+1}. Let k be the largest integer such that $L \subset R^k/R^{m+1}$. We write $\overline{v_{\underline{i}}}$ for the image of $v_{\underline{i}}$ in M and define $\mathrm{rk}(\overline{v_{\underline{i}}})$ as $\mathrm{rk}(v_{\underline{i}})$. We claim that M is the span of the images of monomials. For this is true of $H.v_{\underline{i}}$ for a primitive $v_{\underline{i}}$ because the latter is the span of $x^{\underline{j}} v_{\underline{i}}$ where $x^{\underline{j}}$ runs over the standard basis of H. By part (1) same holds for R^m for every m, hence for M. In fact we have

$$R^m = \bigl(v_{\underline{i}} \bigm| \mathrm{rk}(v_{\underline{i}}) \geqslant m\bigr).$$

Let u be a generator of L written as

$$u = \sum c_{\underline{i}} \overline{v_{\underline{i}}}, \quad 0 \neq c_{\underline{i}} \in \Bbbk. \tag{$*$}$$

By the choice of k the sum u_k of terms of $(*)$ of rank k is nonzero. Let us call the number of terms for u_k in the sum $(*)$ *the length* of u_k. We pick a generator u with u_k of the smallest length. As $k < m \leqslant \ell - 1$ for each $\overline{v_{\underline{i}}}$ of rank k there is $j \in S(\gamma)$ with $i_j < e_j + 1$ where $e_j = e_j(\gamma)$. Set $u' = x_j^{e_j+1-i_j} u$ and observe that $u' \neq 0$, because distinct terms $\overline{v_{\underline{i}}}$ remain distinct or zero upon multiplication by $x_j^{e_j+1-i_j}$ and $x_j^{e_j+1-i_j}\overline{v_{\underline{i}}} \neq 0$, since it has rank $k+1 \leqslant m$. However, u'_k has length smaller than u_k, a contradiction.

(3) By part (1)

$$\mathbb{L}^m = \sum H.\overline{v_{\underline{i}}}, \tag{$**$}$$

where $v_{\underline{i}}$ runs over all primitive basis vectors of rank m. For each $\underline{i}$ the set $B = \{x^{\underline{j}}\overline{v_{\underline{i}}} \mid \operatorname{rk}(x^{\underline{j}} v_{\underline{i}}) = m\}$ is a basis of $H.\overline{v_{\underline{i}}}$. The proof follows immediately, once if we show that $\overline{v_{\underline{i}}}$ is the only primitive vector of $H.\overline{v_{\underline{i}}}$ within a scalar multiple.

Suppose u is a primitive vector of $H.\overline{v_{\underline{i}}}$. Write out u in basis B

$$u = \sum c_{\underline{j}} x^{\underline{j}}\overline{v_{\underline{i}}}, \quad 0 \neq c_{\underline{j}} \in \Bbbk.$$

Since $v_{\underline{i}}$ is primitive, the argument of Lemma 3.4 shows that

$$y_k x^{\underline{j}}\overline{v_{\underline{i}}} = d_{\underline{j}} x^{\underline{j}-u_k}\big(q_k^{j_k-1}\eta(a_k b_k) - 1\big)\overline{v_{\underline{i}}}, \quad d_{\underline{j}} \in \Bbbk^\bullet, \tag{$***$}$$

where η is the weight of $v_{\underline{i}}$. Monomials $x^{\underline{j}-u_k}\overline{v_{\underline{i}}}$ are distinct elements of B, which implies that $y_k u = 0$ if and only if $y_k x^{\underline{j}}\overline{v_{\underline{i}}} = 0$. This condition must hold for all k and therefore, by Eq. $(***)$ it is equivalent to $x^{\underline{j}} \otimes 1_\eta$ is primitive in $Z(\eta)$. From the proof of part (1) we have that for every $k \notin S(\gamma)$, $j_k = 0$, and for $k \in S(\gamma)$, either $j_k = 0$, or $j_k = m_k - e_k(\gamma) - 1, e_k(\gamma) + 1$. Assuming $j_k \neq 0$, in the first case $x^{\underline{j}}\overline{v_{\underline{i}}} = 0$, and in the second $\operatorname{rk}(x^{\underline{j}} v_{\underline{i}}) \geqslant m+1$, hence $x^{\underline{j}}\overline{v_{\underline{i}}} = 0$ again. Thus $\underline{j} = 0$, so that $u = c\overline{v_{\underline{i}}}$ for some $c \in \Bbbk$. It follows that every $H.\overline{v_{\underline{i}}}$ is a simple module and the sum $(**)$ is direct. For otherwise, some primitive $\overline{v_{\underline{i}}}$ would be a linear combination of other primitive monomials of $\mathbb{L}^m$, a contradiction. □

3.3. *The coradical of* H^*

We denote by $\rightharpoonup$ and $\leftharpoonup$ two standard actions of H and H^* on each other [32, Chapter 5]. For every $\gamma \in \widehat{G}$ we define subcoalgebra $C(\gamma)$ by $C(\gamma) = H \rightharpoonup \gamma \leftharpoonup H$.

Proposition 3.8. *The family* $\{C(\gamma) \mid \gamma \in \widehat{G}\}$ *contains every simple subcoalgebra of* H^*. *Thus*

$$\operatorname{corad}(H^*) = \sum_{\gamma \in \widehat{G}} C(\gamma).$$

Proof. It suffices to show that $H \rightharpoonup \gamma \simeq L(\gamma)$. To this end we observe $g \rightharpoonup \gamma = \gamma(g)\gamma$ and $y_k \rightharpoonup \gamma = 0$ for all k. The first of these equalities is obvious. For the second we compute

$$(y_k \rightharpoonup \gamma)\big(x^{\underline{i}} y^{\underline{j}} g\big) = \gamma\big(x^{\underline{i}} y^{\underline{j}} g y_k\big) = \chi_k^{-1}(g)\gamma\big(x^{\underline{i}} y^{\underline{j}} y_k g\big) = 0,$$

by the definition of γ. We see that γ is a primitive vector of weight γ. Therefore $H \rightharpoonup \gamma$ is the image of $Z(\gamma)$ under $\phi : 1 \otimes 1_\gamma \mapsto \gamma$. It remains to show that $\phi(R(\gamma)) = 0$. By Proposition 3.5 this is equivalent to the equality $x_k^{e_k+1} \rightharpoonup \gamma = 0$ for every $k \in S(\gamma)$.

Pick an integer m. By definition of the left action $\upsilon = (x_k^m \rightharpoonup \gamma)(x^{\underline{i}} y^{\underline{j}} g) = \gamma(x^{\underline{i}} y^{\underline{j}} g x_k^m)$. Since every $g \in G$ and every y_j, $j \neq k$, skew commute with x_k we can reduce υ to the form

$$\upsilon = c\chi_k^m(g)\gamma\big(x^{\underline{i}} y^{\underline{j}'}\big(y_k^{j_k} x_k^m\big) y^{\underline{j}''}\big), \quad c \in \Bbbk^\bullet,$$

where $\underline{j}' = (j_1, \dots, j_{k-1})$, $\underline{j}'' = (j_{k+1}, \dots, j_n)$. Using Lemma 1.3 we see readily that $\upsilon = 0$ unless $\underline{i} = \underline{0}$, $\underline{j} = mu_k$. When these conditions hold $\upsilon = \chi_k^m(g)\gamma(f_m^{m,m})\gamma(g)$, where $f_m^{m,m} = (m)_{q_k}! \prod_{p=1}^m (q_k^{m-p} a_k b_k - 1)$ again by Lemma 1.3. As $\prod_{p=1}^m (q^{m-p}\gamma(a_k b_k) - 1) = 0$ for every $k \in S(\gamma)$ and $m \geqslant e_k + 1$, we conclude that every $C(\gamma)$ is simple coalgebra.

On the other hand every simple H-module is isomorphic to $L(\gamma)$ by Proposition 3.5, which completes the proof. $\square$

The functions

$$c_k^m : \widehat{G} \to \Bbbk, \quad c_k^m(\gamma) = \prod_{p=1}^m \big(q_k^{m-p}\gamma(a_k b_k) - 1\big), \tag{3.4}$$

will play a rôle below.

4. The Drinfel'd double

4.1. Multiplication in $D(H)$

The original definition of the Drinfel'd double $D(H)$ [14] of a Hopf algebra is rather technical. For an intrinsic definition of $D(H)$ via the double crossproduct construction see [23,24]. We will follow, though, a more transparent description of $D(H)$ due to Doi and Takeuchi [12].

We recall that $D(H)$ is $H^* \otimes H$ as a vector space and $H^{*\,\mathrm{cop}} \otimes H$ as a coalgebra with the tensor product coalgebra structure. Note that if S is the antipode of H, then $(S^{-1})^*$ is the antipode of $H^{*\,\mathrm{cop}}$. There is a natural bilinear form

$$\tau : H^{*\,\mathrm{cop}} \otimes H \to \Bbbk, \qquad \tau(\alpha, h) = \alpha(h), \quad \text{for } \alpha \in H^*, h \in H.$$

τ is an invertible bilinear form in the convolution algebra $\mathrm{Hom}_{\Bbbk}(H^{*\,\mathrm{cop}} \otimes H, \Bbbk)$ with the inverse $\tau^{-1}(\alpha, h) = \tau((S^{-1})^*(\alpha), h)$. Using τ the algebra structure on $D(H)$ is given with product

$$(\alpha \otimes h)(\beta \otimes k) = \alpha\tau(\beta_3, h_1)(\beta_2 \otimes 1)(1 \otimes h_2)\tau^{-1}(\beta_1, h_3)k, \tag{4.1}$$

where $\Delta_{H^*}^{(2)}(\beta) = \beta_1 \otimes \beta_2 \otimes \beta_3$ and $\Delta_H^{(2)}(h) = h_1 \otimes h_2 \otimes h_3$. In what follows we will drop the "$\otimes$"-sign and write αh. The essential part of definition (4.1) is

$$\begin{aligned} h\beta &= \tau(\beta_3, h_1)\beta_2 h_2 \tau^{-1}(\beta_1, h_3) \\ &= \beta_3(h_1)\beta_2 h_2 \beta_1\big(S^{-1}(h_3)\big). \end{aligned} \tag{4.2}$$

Inverting (4.2) gives the equivalent identity

$$\begin{aligned} \beta h &= \tau^{-1}(\beta_3, h_1) h_2 \beta_2 \tau(\beta_1, h_3) \\ &= \beta_3\big(S^{-1}(h_1)\big) h_2 \beta_2 \beta_1(h_3). \end{aligned} \tag{4.3}$$

It is convenient to rewrite identities (4.2) and (4.3) in terms of actions $\rightharpoonup$ and $\leftharpoonup$. An immediate verification gives

$$\begin{aligned} h\beta &= \big(h_1 \rightharpoonup \beta \leftharpoonup S^{-1}(h_3)\big)h_2 \\ &= \beta_2\big(\big(S^{-1}\big)^*(\beta_1) \rightharpoonup h \leftharpoonup \beta_3\big) \quad \text{and} \end{aligned} \tag{4.4}$$

$$\begin{aligned} \beta h &= h_2\big(S^{-1}(h_1) \rightharpoonup \beta \leftharpoonup h_3\big) \\ &= \big(\beta_1 \rightharpoonup h \leftharpoonup \big(S^{-1}\big)^*(\beta_3)\big)\beta_2. \end{aligned} \tag{4.5}$$

We note that formulas (4.4) and (4.5) were obtained in [27] and [31], respectively.

One consequence of (4.4) is the formula $g\alpha g^{-1} = g \rightharpoonup \alpha \leftharpoonup g^{-1}$. It shows that H^* is invariant under the action by G by conjugation. Therefore we have a Hopf subalgebra $\widetilde{H^*} := H^* \# \Bbbk G$ in $D(H)$. Next, suppose x is an $(a,1)$-primitive element of H satisfying $gx = \chi_x(g)xg$ for $\chi_x \in \widehat{G}$. We associate with x two mappings $\phi_x, \delta_x : \widetilde{H^*} \to \widetilde{H^*}$ defined as follows

$$\begin{aligned} \phi_x(\alpha g) &= \big(a^{-1} \rightharpoonup \alpha\big)\chi_x(g)g, \\ \delta_x(\alpha) &= \big(\alpha \leftharpoonup xa^{-1}\big)a - xa^{-1} \rightharpoonup \alpha \quad \text{and} \\ \delta_x(\alpha g) &= \delta_x(\alpha)\chi_x(g), \end{aligned}$$

for all $\alpha \in H^*$ and $g \in G$.

Lemma 4.1. (1) *ϕ_x is an algebra automorphism and δ_x is a right ϕ_x-derivation of $\widetilde{H^*}$.*

(2) $[x^s,\alpha]_{\phi_x^s} = [x^{s-1},\alpha]_{\phi_x^{s-1}}x + x^{s-1}[x,\phi_x^{s-1}(\alpha)]_\phi$ *for every $\alpha \in H^*$ and $s \geqslant 1$.*

Proof. (1) The claim about ϕ_x is obvious. For the rest it suffices to show that

$$\alpha x = x\phi_x(\alpha) + \delta_x(\alpha) \quad \text{for all } \alpha \in H^*.$$

Note the equalities $\Delta^{(2)}(x) = a \otimes a \otimes x + a \otimes x \otimes 1 + x \otimes 1 \otimes 1$ and $S^{-1}(x) = -xa^{-1}$. Therefore we have from (4.5)

$$\alpha x = a\big(a^{-1} \rightharpoonup \alpha \leftharpoonup x\big) + x\big(a^{-1} \rightharpoonup \alpha\big) - xa^{-1} \rightharpoonup \alpha.$$

As $a(a^{-1} \rightharpoonup \alpha \leftharpoonup x) = (\alpha \leftharpoonup xa^{-1})a$ by (4.4) and $gx = \chi_x(g)xg$, the proof of (1) is complete.

(2) We recall that $[x^s,\alpha]_{\phi_x^s}$ stands for the right derivation $\alpha \mapsto \alpha x^s - x^s\phi_x^s(\alpha)$ as defined in Section 1.2. The proof of the identity is by a direct verification. $\square$

For the sequel we must modify the standard basis of H and the generators of H^*. Since every $g \in \widehat{G}$ skew commutes with all x_i and y_i the set $\{x^{\underline{i}}gy^{\underline{j}} \mid 0 \leqslant i_k, j_k < m_k \text{ and } g \in G\}$ is another basis of H. We define the functionals γ, ξ_k, η_k for all $k \in \underline{n}$ by setting

$$\gamma\big(x^{\underline{i}}gy^{\underline{j}}\big) = \delta_{\underline{0},\underline{i}}\delta_{\underline{0},\underline{j}}\gamma(g), \tag{4.6}$$

$$\xi_k\big(x^{\underline{i}}gy^{\underline{j}}\big) = \delta_{u_k,\underline{i}}\delta_{\underline{0},\underline{j}} \quad \text{for every } g \in G, \tag{4.7}$$

$$\eta_k\big(x^{\underline{i}}gy^{\underline{j}}\big) = \delta_{\underline{0},\underline{i}}\delta_{u_k,\underline{j}} \quad \text{for every } g \in G. \tag{4.8}$$

By an argument almost identical to one for Lemma 2.2 one can show

Lemma 4.2. *The formulas*

$$\xi_k^c \gamma\left(x^{\underline{i}} g y^{\underline{j}}\right) = (c)_{q_k}! \delta_{\underline{i}, c u_k} \delta_{\underline{j}, \underline{0}} \gamma(g) \quad \text{and}$$
$$\eta_k^c \gamma\left(x^{\underline{i}} g y^{\underline{j}}\right) = (c)_{q_k}! \delta_{\underline{i}, \underline{0}} \delta_{\underline{j}, c u_k} \gamma(g).$$

hold for all $\gamma \in \widehat{G}$, $k \in \underline{n}$ *and* $0 \leqslant c \leqslant m_k$.

We also record an analog of Proposition 2.4.

Proposition 4.3.

(1) *For every* $\gamma \in \widehat{G}$ *and* $1 \leqslant k \leqslant n$

$$\gamma \xi_k = \gamma(a_k) \xi_k \gamma \quad \text{and} \quad \gamma \eta_k = \gamma(b_k) \eta_k \gamma.$$

(2) *For all* $s, t \in \underline{n}$ *the equalities*

$$\xi_s \xi_t = \chi_s(a_t) \xi_t \xi_s,$$
$$\eta_s \eta_t = \chi_s(b_t) \eta_t \eta_s,$$
$$\xi_s \eta_t = \eta_t \xi_s$$

hold.

We move on to an explicit description of multiplication in $D(H)$. We start off with the conjugation action of G.

Lemma 4.4. *For all* $g \in G$, $\gamma \in \widehat{G}$, $1 \leqslant k \leqslant n$ *the identities*

$$g\gamma = \gamma g, \tag{4.9}$$
$$g \xi_k = \chi_k^{-1}(g) \xi_k g, \tag{4.10}$$
$$g \eta_k = \chi_k(g) \eta_k g \tag{4.11}$$

hold.

Proof. By (4.2) $g \alpha g^{-1} = g \rightharpoonup \alpha \leftharpoonup g^{-1}$. From the definition of γ, ξ_k, η_k the equations

$$g \rightharpoonup \gamma = \gamma(g)\gamma \quad \text{and} \quad \gamma \leftharpoonup g = \gamma(g)\gamma, \tag{4.12}$$
$$g \rightharpoonup \xi_k = \xi_k \quad \text{and} \quad \xi_k \leftharpoonup g = \chi_k(g)\xi_k, \tag{4.13}$$
$$g \rightharpoonup \eta_k = \chi_k(g)\eta_k \quad \text{and} \quad \eta_k \leftharpoonup g = \eta_k \tag{4.14}$$

follow which complete the proof. $\square$

We need a technical lemma. Below we use the convention that for any set of variables v_j, $v^{\underline{i}} = 0$ if $i_k < 0$ for at least one k.

Lemma 4.5. *There are scalars $c, c', d, d' \in \Bbbk^\bullet$ depending on k, $\underline{i}$, $\underline{j}$ and $g, h \in G$ such that*

$$\big(x^{\underline{i}} g y^{\underline{j}}\big)(x_k h) = c x^{\underline{i}+u_k} g h y^{\underline{j}} + c' x^{\underline{i}} g h \big(q_k^{j_k-1} a_k b_k - 1\big) y^{\underline{j}-u_k}, \tag{4.15}$$

$$(y_k h)\big(x^{\underline{i}} g y^{\underline{j}}\big) = d x^{\underline{i}} g h y^{\underline{j}+u_k} + d' x^{\underline{i}-u_k} g h \big(q_k^{i_k-1} a_k b_k - 1\big) y^{\underline{j}}. \tag{4.16}$$

Proof. Since elements of G and $y_l, l \neq k$, skew commute with x_k, $(x^{\underline{i}} g y^{\underline{j}})(x_k h) = a x^{\underline{i}} g h y^{\underline{j}'} y_k^{j_k} x_k y^{\underline{j}''}$ with $\underline{j}' = (j_1, \ldots, j_{k-1})$ and $\underline{j}'' = (j_{k+1}, \ldots, j_n)$. Next we use Lemma 1.2(1) according to which

$$y_k^{j_k} x_k = q_k^{j_k} x_k y_k^{j_k} - q_k (j_k)_{q_k} \big(q_k^{j_k-1} a_k b_k - 1\big) y_k^{j_k-1}.$$

This formula and the fact that x_k skew commutes with $x_l, l \neq k$, completes the proof of (4.15).

The proof of (4.16) is almost identical. One must use Lemma 1.2(2) together with the observation that $x_l, l \neq k$, commutes with $a_k b_k$. □

The next three lemmas completely determine the algebra structure of $D(H)$.

Lemma 4.6. *For every $\gamma \in \widehat{G}$ and $1 \leqslant k \leqslant n$*

$$\gamma x_k = \gamma\big(a_k^{-1}\big) x_k \gamma + \gamma\big(a_k^{-1}\big) q_k \big(\gamma(a_k b_k) - 1\big) \eta_k \gamma, \tag{4.17}$$

$$\gamma y_k = \gamma\big(b_k^{-1}\big) y_k \gamma - \gamma\big(b_k^{-1}\big)\big(\gamma(a_k b_k) - 1\big) \xi_k \gamma b_k. \tag{4.18}$$

Proof. By Lemma 4.1 we need only to compute $\delta_{x_k}(\gamma)$. By definition this involves finding $\gamma \leftharpoonup x_k a_k^{-1}$ and $x_k a_k^{-1} \rightharpoonup \gamma$. First we show that

$$\gamma \leftharpoonup x_k a_k^{-1} = 0. \tag{4.19}$$

For by definition $(\gamma \leftharpoonup x_k a_k^{-1})(x^{\underline{i}} g y^{\underline{j}}) = \gamma(x_k a_k^{-1} x^{\underline{i}} g y^{\underline{j}})$ and the latter is zero because $x_k a_k^{-1} x^{\underline{i}} g y^{\underline{j}} = c x^{\underline{i}+u_k} g a_k^{-1} y^{\underline{j}}, c \in \Bbbk^\bullet$, with $\underline{i} + u_k \neq \underline{0}$ for all $\underline{i}$, $\underline{j}$, g.

Next we compute

$$\upsilon := \big(x_k a_k^{-1} \rightharpoonup \gamma\big)\big(x^{\underline{i}} g y^{\underline{j}}\big) = \gamma\big(x^{\underline{i}} g y^{\underline{j}} x_k a_k^{-1}\big).$$

Using (4.15) we express $\upsilon = \upsilon_1 + \upsilon_2$ where $\upsilon_1 = c\gamma(x^{\underline{i}+u_k} g a_k^{-1} y^{\underline{j}})$ and $\upsilon_2 = c'\gamma(x^{\underline{i}} g a_k^{-1} (q_k^{j_k-1} a_k b_k - 1) y^{\underline{j}-u_k})$. As in the proof of (4.19) $\upsilon_1 = 0$ for all basis elements. The definition of γ makes it clear that $\upsilon_2 = 0$, unless $\underline{i} = \underline{0}$ and $\underline{j} = u_k$. In the latter case $c' = -q_k$, hence $\upsilon_2 = -q_k \gamma(g a_k^{-1}(a_k b_k - 1))$. That is to say

$$x_k a_k^{-1} \rightharpoonup \gamma = -q_k \gamma\big(a_k^{-1}\big) \gamma(a_k b_k - 1) \eta_k \gamma, \tag{4.20}$$

by Lemma 4.2, and this completes the proof of (4.17).

The proof of (4.18) is almost identical. The main steps are the equalities

$$y_k b_k^{-1} \rightharpoonup \gamma = 0 \quad \text{and} \quad \gamma \leftharpoonup y_k b_k^{-1} = -\gamma\big(b_k^{-1}\big) \gamma(a_k b_k - 1) \xi_k \gamma. \qquad \square \tag{4.21}$$

Lemma 4.7. *For all $k, l \in \underline{n}$*

$$\xi_l x_k = x_k \xi_l + \delta_{k,l}(a_l - \chi_k), \tag{4.22}$$

$$\xi_l y_k = y_k \xi_l - \delta_{k,l} q_k^{-1}(q_k - 1)\xi_k^2 b_k. \tag{4.23}$$

Proof. We use again Lemma 4.1. Since $a_k^{-1} \rightharpoonup \xi_l = \xi_l$ by (4.10), it remains to find $\delta_{x_k}(\xi_l)$. First, we claim that

$$\xi_l \leftharpoonup x_k a_k^{-1} = \delta_{k,l}\epsilon. \tag{4.24}$$

This is the matter of showing that $\upsilon = \xi_l(x_k a_k^{-1} x^{\underline{i}} g y^{u\underline{j}}) = \delta_{k,l}\delta_{\underline{i},\underline{0}}\delta_{\underline{j},\underline{0}}$. We observe that $\upsilon = c\chi^{\underline{i}-\underline{j}}(a_k^{-1}) \times \xi_l(x^{\underline{i}+u_k} h y^{\underline{j}})$ for some $h \in G$ and $0 \neq c \in \Bbbk$ with $c = 1$ when $\underline{i} = \underline{0} = \underline{j}$, which yields the claim by the definition of ξ_l.

Next we show the identity

$$x_k a_k^{-1} \rightharpoonup \xi_l = \delta_{k,l}\chi_k. \tag{4.25}$$

Now we must compute $\upsilon' = \xi_l(x^{\underline{i}} g y^{\underline{j}} x_k a_k^{-1})$. Applying the straightening out formula (4.15) we write $\upsilon' = \upsilon_1 + \upsilon_2$ where $\upsilon_1 = c\xi_l(x^{\underline{i}+u_k} g a_k^{-1} y^{\underline{j}})$ and $\upsilon_2 = c'\xi_l(x^{\underline{i}} g a_k^{-1}(q_k^{j_k-1} a_k b_k - 1) y^{\underline{j}-u_k})$. We note that υ_2 is zero for all $\underline{i}, g$ and $\underline{j}$. For, if $\underline{j} - u_k \neq \underline{0}$, then surely $\upsilon_2 = 0$. Else, $\upsilon_2 = c'\xi_l(x^{\underline{i}}(s - t))$ for some $s, t \in G$, which is again zero.

As for υ_1, if $l \neq k$, then $\upsilon_1 = 0$, because $\underline{i} + u_k \neq u_l$ for all $\underline{i}$. Suppose $l = k$. Then $\upsilon_1 \neq 0$ if and only if $\underline{i} = \underline{0} = \underline{j}$. If so, $c = \chi_k(g)$, hence $\upsilon_1 = \chi_k(g)\delta_{\underline{i},\underline{0}}\delta_{\underline{j},\underline{0}}$, which gives (4.25).

For the proof of the second part we need two observations. First off, the equality $y_k b_k^{-1} \rightharpoonup \xi_l = 0$ for all k, l is self-evident. In the second place we claim the identity

$$\xi_l \leftharpoonup y_k b_k^{-1} = -q_k^{-1}(q_k - 1)\xi_k^2. \tag{4.26}$$

Again the proof boils down to finding $\upsilon = \xi_l(y_k b_k^{-1} x^{\underline{i}} g y^{\underline{j}})$ which by (4.16) splits up as $\upsilon = \upsilon_1 + \upsilon_2$ with $\upsilon_1 = d\xi_l(x^{\underline{i}} g b_k^{-1} y^{\underline{j}+u_k})$ and $\upsilon_2 = d'\xi_l(x^{\underline{i}-u_k} g b_k^{-1}(q_k^{i_k-1} a_k b_k - 1) y^{\underline{j}})$. Now υ_1 is always zero, because $\underline{j} + u_k \neq u_l$ for all $\underline{j}$. If $l \neq k$, then $\underline{i} - u_k \neq u_l$ for all $\underline{i}$ forces $\upsilon_2 = 0$ for all choices of $\underline{i}, g, \underline{j}$. Take the case $k = l$. Now $\upsilon_2 = 0$ for all $\underline{i}, \underline{j}$ such that $\underline{i} - u_k \neq u_k$ or $\underline{j} \neq \underline{0}$. In the remaining case, i.e. $\underline{i} = 2u_k$ and $\underline{j} = \underline{0}$, we have from Lemma 1.2(2) the identity

$$y_k b_k^{-1} x_k^2 = q_k^{-2} y_k x_k^2 b_k^{-1} = \bigl(x_k^2 y_k - q_k^{-1}(2)_{q_k} x_k(q_k a_k b_k - 1)\bigr) b_k^{-1},$$

which gives $d' = -q_k^{-1}(2)_{q_k}$. Noting that $\xi_k(x_k(q_k s - t)) = q_k - 1$ for all $s, t \in G$ we arrive at $\upsilon_2 = -q_k^{-1}(2)_{q_k}(q_k - 1)\delta_{\underline{i},2u_k}\delta_{\underline{j},\underline{0}}$. In view of Lemma 4.2 we obtain the desired formula. □

Lemma 4.8. *For all $k, l \in \underline{n}$*

$$\eta_l x_k = q_{kl}^{-1} x_k \eta_l + \delta_{k,l}(q_k - 1)\eta_k^2, \tag{4.27}$$

$$\eta_l y_k = q_{lk}^{-1} y_k \eta_l + \delta_{k,l} q_k^{-1}(\chi_k b_k - \epsilon). \tag{4.28}$$

Proof. The proof follows from the following equations

$$\eta_l \leftharpoonup x_k a_k^{-1} = 0 \quad \text{for all } l, k, \quad \text{and} \quad x_k a_k \rightharpoonup \eta_l = \delta_{k,l}(q_k - 1)\eta_k^2, \tag{4.29}$$

$$y_k b_k^{-1} \rightharpoonup \eta_l = \delta_{k,l} q_k^{-1} \epsilon \quad \text{and} \quad \eta_l \leftharpoonup y_k b_k^{-1} = \delta_{k,l} q_k^{-1} \chi_k. \tag{4.30}$$

A verification of these equations follows from the proof of the preceding lemma and is left for the reader. □

In keeping with our convention we put $\underline{m} = (m_1, \dots, m_n)$. We let $\mathbb{Z}(\underline{m})$ denote $[m_1] \times \cdots \times [m_n]$. As a consequence of the last three lemmas we have the fact that the set of all products of $g\gamma, x^{\underline{i}}, y^{\underline{j}}, \xi^{\underline{k}}, \eta^{\underline{l}}$ with $g\gamma \in G \times \widehat{G}$ and $\underline{i}, \underline{j}, \underline{k}, \underline{l} \in \mathbb{Z}(\underline{m})$ in any prescribed order forms a basis for $D(H)$.

4.2. Parametrization of simple $D(H)$-modules

We denote by Γ the group $G \times \widehat{G}$. We recall that $J(A)$ denotes the radical of an algebra A. We consider the subalgebra $A = A(\Gamma, y, \xi)$ of $D(H)$ generated by Γ, y_i, ξ_i for all $i \in \underline{n}$. Relations (4.18) and (4.23) imply that A is a free span of the set $\{y^{\underline{i}}\xi^{\underline{j}}g\gamma \mid 0 \leqslant i_k, j_k \leqslant m_k - 1, g\gamma \in \Gamma\}$. We set $|\underline{i}| = \sum i_k$ for an n-tuple $\underline{i}$.

Lemma 4.9. *A is a right coideal of $D(H)$ and a basic algebra in the sense that*

$$A = \Bbbk\Gamma \oplus J(A).$$

Proof. We let B denote the subalgebra of A generated by Γ and all ξ_k. Clearly $B = \Bbbk\Gamma \oplus J(B)$, where $J(B)$ is the span of all $\xi^{\underline{j}}g\gamma$ with $|\underline{j}| > 0$. Since the elements of Γ skew commute with every ξ_k, $J(B)$ is nilpotent. Let N be the largest integer with $J(B)^N \neq 0$. Set I to be the span of all $y^{\underline{i}}\xi^{\underline{j}}g\gamma$ with $|\underline{i}| + |\underline{j}| > 0$. Evidently I is a complement of $\Bbbk\Gamma$ in A, and $I = \sum_{|\underline{i}|>0} y^{\underline{i}}B + J(B)$. We lift the radical filtration

$$B \supset J(B) \supset \cdots \supset J^N(B) \supset 0,$$

to a filtration

$$I = I_0 \supset I_1 \supset \cdots \supset I_N \supset 0,$$

where I_k is defined by $I_k = \sum_{|\underline{i}|>0} y^{\underline{i}} J^k(B) + J^k(B)$. Thanks to the relations (4.18) and (4.23) the elements of B skew commute with all $y^{\underline{i}}z$, $z \in J^k(B)$, modulo I_{k+1}. Therefore the I_k form an ideal filtration with nilpotent quotients I_k/I_{k+1} because the y_k generate a nilpotent subalgebra. Thus $I = J(A)$.

We take up the first claim. The comultiplication of $y^{\underline{i}}g$ (cf. (2.4)) makes it clear that it suffices to show that the subalgebra B' of H^* generated by $\widehat{G}$ and the ξ_k is a right coideal. Since $\Delta_D = \Delta_{H^{*\mathrm{cop}}}$ on H^* this is equivalent to B' is a left coideal of H^*, or a right H-submodule with respect to the '$\leftharpoonup$'-action. As H^* is an H-module algebra we need only to establish inclusion $z \leftharpoonup h \in B'$ for z and h running over the generators of B' and H, respectively. Now using (4.12), (4.13), and (4.19), (4.21) we have

$$\gamma \leftharpoonup g = \gamma(g)\gamma, \qquad \xi_k \leftharpoonup g = \chi_k(g)\xi_k \quad \text{for all } g \in G, \quad \text{and}$$

$$\gamma \leftharpoonup x_k = 0, \quad \text{and} \quad \gamma \leftharpoonup y_k = -\big(\gamma(a_k b_k) - 1\big)\xi_k\gamma.$$

A reference to (4.24) and (4.26) completes the proof. □

The above lemma makes it obvious that every simple A-module is 1-dimensional. Pick $\lambda \in \widehat{\Gamma}$ and let $\Bbbk_\lambda$ denote the A-module $\Bbbk$ with the A-action defined by

$$J(A).1_{\Bbbk} = 0 \quad \text{and} \quad g\gamma.1_{\Bbbk} = \lambda(g\gamma).$$

We write 1_λ for the element $1_{\Bbbk}$ of $\Bbbk_\lambda$. We define a family of D-modules $I(\lambda), \lambda \in \widehat{\Gamma}$ by setting

$$I(\lambda) = D \otimes_A \Bbbk_\lambda,$$

where we write $D = D(H)$. As we mentioned earlier the set $\{x^{\underline{i}}\eta^{\underline{j}}y^{\underline{k}}\xi^{\underline{l}}g\gamma \mid g\gamma \in \Gamma, \underline{i}, \underline{j}, \underline{k}, \underline{l} \in \mathbb{Z}(\underline{m})\}$ is a basis for $D(H)$. Therefore, directly from the definition we obtain that $I(\lambda)$ is a free span of the set $\{x^{\underline{i}}\eta^{\underline{j}} \otimes 1_\lambda \mid \underline{i}, \underline{j} \in \mathbb{Z}(\underline{m})\}$. This is *the standard basis* of $I(\lambda)$.

Proposition 4.10. *For every $\lambda \in \widehat{\Gamma}$ the D-module $I(\lambda)$ has a unique maximal submodule.*

Proof. Let Π be the hyperplane of $I(\lambda)$ spanned by all $\{x^{\underline{i}}\eta^{\underline{j}} \otimes 1_\lambda\}$ with $|\underline{i}| + |\underline{j}| > 0$. We claim that every proper D-submodule M of $I(\lambda)$ lies in Π. If not, then

$$1 \otimes 1_\lambda + \sum_{|\underline{i}|+|\underline{j}|>0} c_{\underline{i},\underline{j}} x^{\underline{i}}\eta^{\underline{j}} \otimes 1_\lambda \in M \quad \text{for some } c_{\underline{i},\underline{j}} \in \Bbbk.$$

By an argument verbatim to one used in the proof of Lemma 4.9 the x_i, η_j generate a nilpotent subalgebra of D. Therefore $z = \sum_{|\underline{i}|+|\underline{j}|>0} c_{\underline{i},\underline{j}} x^{\underline{i}}\eta^{\underline{j}} \otimes 1_\lambda$ is nilpotent, hence $1 \otimes 1_\lambda \in M$, a contradiction. $\square$

Let $R(\lambda)$ denote the radical of $I(\lambda)$, i.e. the sum of all proper submodules or zero, if $I(\lambda)$ is simple. We set

$$L(\lambda) = I(\lambda)/R(\lambda).$$

Let M be a D-module. An element $0 \neq v \in M$ is called *weight element* of weight $\mu \in \widehat{\Gamma}$ if

$$g\gamma.v = \mu(g\gamma)v \quad \text{holds for all } g\gamma \in \Gamma.$$

We say that $v \neq 0$ is *primitive* if $J(A).v = 0$. We note that every D-module M contains primitive weight elements, in fact, a simple submodule of the A-socle of M is spanned by a primitive element.

The subalgebra of $D(H)$ generated by H^* and x_k is an Ore extension with the automorphism $\phi_{x_k} : \alpha \mapsto (a_k^{-1} \rightharpoonup \alpha)$, $\alpha \in H^*$, and a right ϕ_{x_k}-derivation δ_{x_k} determined on generators of H^* by relations (4.17), (4.22) and (4.27). Therefore for every $\alpha \in H^*$, $\alpha x_k^s = \sum_{i=0}^s x_k^i \alpha_i$ for some $\alpha_i \in H^*$. The next two lemmas give a more precise form of these identities.

Lemma 4.11. *There are polynomials $h_i^{(s)}(t)$ for $s = 1, 2, \ldots; i = 1, 2, \ldots s$, such that*

$$[x_k^s, \eta_k]_{\phi_{x_k}^s} = \sum_{i=1}^{s} h_i^{(s)}(q_k) x_k^{s-i} \eta_k^{i+1}. \tag{4.31}$$

Proof. To simplify notation we drop the subscript k and set $\phi = \phi_{x_k}$. We induct on s noting that the case $s = 1$ holds by (4.27). First we have $\phi^s(\eta) = a^{-s} \rightharpoonup \eta = q^{-s}\eta$ by (4.14). Now the induction hypothesis and Lemma 4.1(2) give the identity

$$[x^{s+1}, \eta]_{\phi^{s+1}} = q^{-s}x^s[x, \eta]_\phi + \left(\sum_{i=1}^{s} h_i^{(s)}(q) x^{s-i}\eta^{i+1}\right)x.$$

It remains to pass η^{i+1} over x. We have by Lemma 4.1(1) that $\eta^{i+1}x = q^{-(i+1)}x\eta^{i+1} + \delta_x(\eta^{i+1})$. There δ_x is a right ϕ-derivation with $\delta_x(\eta) = (q-1)\eta^2$ by (4.27). We claim that for every m

$$\delta_x\left(\eta^m\right) = (q-1)(m)_{q^{-1}}\eta^{m+1}. \tag{4.32}$$

For by (1.15) $\delta_x(\eta^{m+1}) = \delta_x(\eta^m)q^{-1}\eta + \eta^m(q-1)\eta^2$, and assuming the formula for m we get $\delta_x(\eta^{m+1}) = (q-1)(q^{-1}(m)_{q^{-1}} + 1)\eta^{m+2}$ as asserted. $\square$

Lemma 4.12. *There are polynomials $g_i^{(s)}(t)$ for $s = 1, 2, \ldots;\ i = 1, 2, \ldots s$, such that*

$$\left[x_k^s, \gamma\right]_{\phi^s_{x_k}} = \gamma\left(a_k^{-s}\right)\left(x_k^s + \sum_{i=1}^{s} g_i^{(s)}(q_k)c_k^i(\gamma)x_k^{s-i}\eta_k^i\right)\gamma, \tag{4.33}$$

where $c_k^i(\gamma)$ are functions defined in (3.4).

Proof. We suppress the subscript k as in the preceding lemma. We argue by induction on s starting with (4.17). Since $\phi^s(\gamma) := a^{-s} \rightharpoonup \gamma = \gamma(a^{-s})\gamma$, Lemma 4.1 gives

$$\begin{aligned}\left[x^{s+1}, \gamma\right]_{\phi^{s+1}} &= x^s\gamma\left(a^{-s}\right)[x, \gamma]_\phi + \left[x^s, \gamma\right]_{\phi^s}x \\ &= \gamma\left(a^{-(s+1)}\right)qc^1(\gamma)x^s\eta\gamma + \gamma\left(a^{-s}\right)\left(\sum_{i=1}^{s} g^i(q)c^i(\gamma)x^{s-i}\eta^i\gamma\right)x,\end{aligned}$$

the last equality by (4.17) and the induction hypothesis. The proof will be completed if we show that

$$\left(\eta^i\gamma\right)x = \gamma\left(a^{-1}\right)\left(\kappa_1 x\eta^i\gamma + \kappa_2\left(q^i\gamma(ab) - 1\right)\eta^{i+1}\gamma\right),$$

where κ_i are some polynomials of q. This equality is derived as follows. First, Lemma 4.1 lets us write $(\eta^i\gamma)x = x(a^{-1} \rightharpoonup \eta^i\gamma) + \delta_x(\eta^i\gamma)$. Next, the '$\rightharpoonup$'-action is an algebra homomorphism, hence $a^{-1} \rightharpoonup \eta^i\gamma = q^{-i}\gamma(a^{-1})\eta^i\gamma$ with the help of Lemma 4.4. Lastly, recalling that δ_x is a right ϕ-derivation we compute

$$\begin{aligned}\delta_x\left(\eta^i\gamma\right) &= \delta_x\left(\eta^i\right)\phi(\gamma) + \eta^i\delta_x(\gamma) \\ &= (q-1)(i)_{q^{-1}}\eta^{i+1}\gamma\left(a^{-1}\right)\gamma + \gamma\left(a^{-1}\right)qc^1\eta^{i+1}\gamma \\ &= \gamma\left(a^{-1}\right)\left[(q-1)(i)_{q^{-1}} + q\left(\gamma(ab) - 1\right)\right]\eta^{i+1}\gamma,\end{aligned}$$

where the second line is written by (4.32) and the basic relation (4.17). It remains to note that the expression in the square brackets equals $q^{-(i-1)}(q^i\gamma(ab) - 1)$. $\square$

We record for the future reference

$$\delta_{x_k}\left(\eta_k^i\gamma\right) = q_k^{-(i-1)}\gamma\left(a_k^{-1}\right)\left(q_k^i\gamma(a_k b_k) - 1\right)\eta_k^{i+1}\gamma. \tag{4.34}$$

We move on to the general case of the preceding lemma. For an n-tuple $\underline{i}$ we write $c_{\underline{i}} = c_1^{i_1}\cdots c_n^{i_n}$. We recall that $a^{\underline{s}}$ stands for $a_1^{s_1}\cdots a_n^{s_n}$.

Lemma 4.13. *For every pair* $(\underline{s},\underline{t})$ *with* $\underline{s},\underline{t}\in\mathbb{Z}(\underline{m})$ *there are polynomials* $g_{\underline{i}}=g_{\underline{i}}(q_1,\dots,q_n)$, $\underline{i}\leqslant\underline{s}$, *such that*

$$\gamma\left(x^{\underline{s}}\eta^{\underline{t}}\right)\otimes 1_\lambda=\lambda(\gamma)\gamma\left(a^{-\underline{s}}b^{\underline{t}}\right)\left(x^{\underline{s}}\eta^{\underline{t}}\otimes 1_\lambda+\sum_{\underline{i}\leqslant\underline{s}}g_{\underline{i}}c_{\underline{i}}(\gamma)x^{\underline{s}-\underline{i}}\eta^{\underline{t}+\underline{i}}\otimes 1_\lambda\right). \tag{4.35}$$

Proof. We derive from the previous lemma that $\gamma(x^{\underline{s}}\eta^{\underline{t}})\otimes 1_\lambda$ is the sum of monomials $m_{\underline{i}}=\gamma(a^{-\underline{s}})g_1^{i_1}\cdots g_n^{i_n}c_{\underline{i}}(\gamma)x_1^{s_1-i_1}\eta_1^{i_1}\cdots x_n^{s_n-i_n}\eta_n^{i_n}\gamma\eta^{\underline{t}}$. Now observe that η_i skew commutes with every x_j, $j\neq i$, all η_j, and $\gamma\eta^{\underline{t}}=\gamma(b^{\underline{t}})\eta^{\underline{t}}$ by Proposition 4.3(1). Thus $m_{\underline{i}}$ can be rewritten as $q_1^{p_1}\cdots q_n^{p_n}x^{\underline{s}-\underline{i}}\eta^{\underline{t}+\underline{i}}$ for a suitable integers p_i and the lemma follows. □

We begin to build the weight space decomposition of $I(\lambda)$. For $g\in G$ we denote by $\hat{g}$ the character of $\widehat{G}$ sending γ to $\gamma(g)$ for every $\gamma\in\widehat{G}$. For $\underline{s},\underline{t}\in(\mathbb{Z})^n$ and $\lambda\in\Gamma$ we define the character $\lambda_{\underline{s},\underline{t}}$ by

$$\lambda_{\underline{s},\underline{t}}=\lambda\widehat{a^{-\underline{s}}b^{\underline{t}}}\chi^{\underline{s}+\underline{t}}.$$

Recall the idempotent $e_\lambda=|\Gamma|^{-1}\sum_{g\gamma}\lambda^{-1}(g\gamma)g\gamma$ associated to $\lambda\in\Gamma$. We denote by $e_{\underline{s},\underline{t}}$ the idempotent corresponding to $\lambda_{\underline{s},\underline{t}}$. We define vector $v_{\underline{s},\underline{t}}$ by the formula $v_{\underline{s},\underline{t}}=e_{\underline{s},\underline{t}}(x^{\underline{s}}\eta^{\underline{t}}\otimes 1_\lambda)$. In the same spirit we let $I_{\underline{s},\underline{t}}(\lambda)$ denote the weight space $e_{\underline{s},\underline{t}}I(\lambda)$. We put an equivalence relation on the set $\mathbb{Z}(\underline{m})\times\mathbb{Z}(\underline{m})$ by declaring $(\underline{s},\underline{t})\sim(\underline{s}',\underline{t}')$ if and only if $\lambda_{\underline{s},\underline{t}}=\lambda_{\underline{s}',\underline{t}'}$. We let $[\underline{s},\underline{t}]$ denote the equivalence class of $(\underline{s},\underline{t})$.

Lemma 4.14.

(1) *The set*

$$\left\{v_{\underline{s},\underline{t}}\;\middle|\;\underline{s},\underline{t}\in\mathbb{Z}(\underline{m})\right\},$$

is a basis for $I(\lambda)$.

(2) *The set*

$$\left\{v_{\underline{s}',\underline{t}'}\;\middle|\;(\underline{s}',\underline{t}')\in[\underline{s},\underline{t}]\right\},$$

is a basis for $I_{\underline{s},\underline{t}}(\lambda)$.

(3) $I(\lambda)=\bigoplus\{I_{\underline{s},\underline{t}}\mid\underline{s},\underline{t}\in\mathbb{Z}(\underline{m})\}$.

Proof. (1) The defining relations of H and (4.11) make it clear that $x^{\underline{s}}\eta^{\underline{t}}\otimes 1_\lambda$ has G-weight $\lambda|_G\chi^{\underline{s}+\underline{t}}$. This observation combined with the previous lemma shows that

$$v_{\underline{s},\underline{t}}=x^{\underline{s}}\eta^{\underline{t}}\otimes 1_\lambda+\sum_{\underline{i}<\underline{s}}g_{\underline{i}}\overline{c_{\underline{i}}}x^{\underline{s}-\underline{i}}\eta^{\underline{t}+\underline{i}}\otimes 1_\lambda, \tag{4.36}$$

where $\overline{c_{\underline{i}}}:=|G|^{-1}\sum_{\gamma\in\widehat{G}}c_{\underline{i}}(\gamma)$ is the average value of $c_{\underline{i}}$ over $\widehat{G}$. We see that $v_{\underline{s},\underline{t}}$ has the leading term $x^{\underline{s}}\eta^{\underline{t}}\otimes 1_\lambda$. We claim that the set in part (1) is linearly independent. Else, there is a linear relation $(*)$ $\sum\kappa_{\underline{s},\underline{t}}v_{\underline{s},\underline{t}}=0$, $0\neq\kappa_{\underline{s},\underline{t}}\in\mathbb{k}$. Pick $(\underline{s}',\underline{t}')$ such that $\underline{s}'$ is the largest among all $(\underline{s},\underline{t})$ involved in $(*)$ and $\underline{t}'$ is the smallest among all $(\underline{s}',\underline{t})$ involved in $(*)$ in the ordering of Section 2.1. Then $x^{\underline{s}'}\eta^{\underline{t}'}\otimes 1_\lambda$ cannot get cancelled, a contradiction. As $\dim I(\lambda)=\operatorname{card}|\mathbb{Z}(\underline{m})|^2$, the assertion holds.

(2) and (3). We note that every $v_{\underline{s}',\underline{t}'}$ with $(\underline{s}',\underline{t}')\in[\underline{s},\underline{t}]$ lies in $I_{\underline{s},\underline{t}}(\lambda)$ by the definition of the latter, hence $\dim I_{\underline{s},\underline{t}}(\lambda)\geqslant|[\underline{s},\underline{t}]|$. However, the sum of cardinalities of the distinct sets $[\underline{s},\underline{t}]$ equals $\dim I(\lambda)$. On the other hand, sum of $I_{\underline{s},\underline{t}}(\lambda)$ is direct of dimension no greater than $\dim I(\lambda)$. This proves (2) and (3). □

The next theorem is the main result of this section. We note that the theorem and its proof resemble theorems of Curtis [11] and Lusztig [21] on parametrization of simple modules.

Theorem 4.15. *The simple modules $L(\lambda)$, $\lambda \in \widehat{\Gamma}$, are a full set of representatives of simple D-modules.*

Proof. As every simple D-module L is generated by a primitive weight element, L is the image of $I(\lambda)$ for a suitable λ. It remains to show that $L(\lambda) \simeq L(\mu)$ implies $\lambda = \mu$. Let Π be the hyperplane in $I(\lambda)$ of Proposition 4.10, and denote by P its image in $L(\lambda)$. Were $L(\lambda) \simeq L(\mu)$, there would be a primitive vector v of weight μ in $L(\lambda)$.

We note that P is a proper subspace of $L(\lambda)$. For, a linear relation

$$1 \otimes 1_\lambda + \sum c_{\underline{i},\underline{j}} x^{\underline{i}} \eta^{\underline{j}} \otimes 1_\lambda \equiv 0 \mod R(\lambda),$$

implies $1 \otimes 1_\lambda \in R(\lambda)$, because the x_l, η_k generate a nilpotent subalgebra, a contradiction. As $\lambda \neq \mu$ by assumption, v is a linear combination of the images of $v_{\underline{s},\underline{t}}$ with $(\underline{s},\underline{t}) \neq (\underline{0},\underline{0})$. By (4.36) $v \in P$, hence, as v is primitive, $L(\lambda)$ is the span of all $x^{\underline{i}} \eta^{\underline{j}} v$. Referring to Lemma 4.11 we see that P is invariant under multiplication by η_k and, of course, x_l. Thus $L(\lambda) \subset P$, a contradiction. $\square$

4.3. The Loewy filtration of $I(\lambda)$

4.3.1. Action of generators on standard basis

Our ultimate goal is to describe the action of generators of D on the weight basis of $I(\lambda)$. The next proposition is a key step toward this goal. It runs smoothly under a certain restriction on the datum for H. We do not know whether this restriction is essential for our results.

For an even m we put $m' = m/2$.

Definition 4.16. We say that a simply linked datum $\mathcal{D}$ is *half-clean* if G does not have nontrivial relations $r = 1$ of the form

$$r = \prod_{i=1}^{n} (a_i b_i)^{t_i},$$

with $t_i = 0$ or m_i'.

The above definition has the following implication for the set of relations of G. G does not have relators $r = \prod_{i=1}^{n} (a_i b_i)^{t_i} \neq 1$ with $t_i < m_i$ for all i. For, since $\chi_j(a_i b_i) = 1$ for every $j \neq i$, $\chi_i(r) = \chi_i(a_i b_i)^{t_i} = q_i^{2t_i}$. Therefore $r = 1$ implies $2t_i \equiv 0 \mod m_i$. Thus $t_i = 0$ if m_i is odd, and $t_i = 0$ or m_i', otherwise.

Proposition 4.17. *Suppose $\mathcal{D}$ is a half-clean datum, $\underline{s} \in \mathbb{Z}(\underline{m})$ and $\underline{t} \in \mathbb{Z}^n$.*

(1) *For every $\underline{i}$, $\underline{0} < \underline{i} \leqslant \underline{s}$,*

$$e_{\underline{s},\underline{t}} \left(x^{\underline{s}-\underline{i}} \eta^{\underline{t}+\underline{i}} \otimes 1_\lambda \right) = 0.$$

(2) *For every $k \in \underline{n}$ and every $\underline{i}$, $\underline{0} \leqslant \underline{i} \leqslant \underline{s}$,*

$$e_{\underline{s}+u_k,\underline{t}} \left(x^{\underline{s}-\underline{i}} \eta^{\underline{t}+u_k+\underline{i}} \otimes 1_\lambda \right) = 0.$$

(3) *For every $i, 0 \leqslant i \leqslant s_k - 1$,*

$$e_{\underline{s},\underline{t}-u_k} \left(x^{\underline{s}-(i+1)u_k} \eta^{\underline{t}+iu_k} \otimes 1_\lambda \right) = 0.$$

Proof. As a preliminary to the proof we point out that for every abelian group G, $\sum_{\gamma\in\widehat{G}}\gamma(g)=|G|\delta_{1,g}$.

(1) Put $v=x^{\underline{s}-\underline{i}}\eta^{\underline{t}+\underline{i}}\otimes 1_\lambda$. By Lemma 4.13 and (4.11) v acquires weight $\lambda_{\underline{s}-\underline{i},\underline{t}+\underline{i}}$ upon multiplication by $g\gamma$. One can see readily that $\lambda_{\underline{s},\underline{t}}^{-1}\lambda_{\underline{s}-\underline{i},\underline{t}+\underline{i}}=\widehat{(ab)^{\underline{i}}}$. Consequently Lemma 4.13 gives the equality

$$e_{\underline{s},\underline{t}}.v=\sum_{\underline{j}\leqslant\underline{s}-\underline{i}} g_{\underline{j}}\left\{|G|^{-1}\sum_{\gamma\in\widehat{G}}\gamma\big((ab)^{\underline{i}}\big)c_{\underline{j}}(\gamma)\right\}x^{\underline{s}-\underline{i}-\underline{j}}\eta^{\underline{t}+\underline{i}+\underline{j}}\otimes 1_\lambda. \tag{$*$}$$

We begin to work out the inner sum in $(*)$. First, for a positive integer m and every $k\in\underline{n}$ we have the expansion

$$c_k^m(\gamma)=\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{l}_{q_k}q_k^{\binom{l}{2}}\gamma\big((a_kb_k)^l\big),$$

by the Gauss' binomial formula [17].

Replacing m by j_k and l by l_k and letting k run over $\underline{n}$ we see that the inner sum in $(*)$ equals $|G|^{-1}\sum_{\underline{l}\leqslant\underline{j}}\gamma((ab)^{\underline{i}+\underline{l}})$. Since $\underline{i}+\underline{l}\leqslant\underline{s}<\underline{m}$ our assumption on $\mathcal{D}$ imply that $(ab)^{\underline{i}+\underline{l}}\neq 1$ for every $\underline{l}$, whence the sum vanishes by the opening remark, and this proves (1).

(2) We set $v=x^{\underline{s}-\underline{i}}\eta^{\underline{t}+u_k+\underline{i}}\otimes 1_\lambda$. A simple verification gives $\lambda^{-1}_{\underline{s}+u_k,\underline{t}}\lambda_{\underline{s}-\underline{i},\underline{t}+u_k+\underline{i}}=\widehat{(ab)^{\underline{i}+u_k}}$. Repeating the argument leading up to the equality $(*)$ we derive that

$$e_{\underline{s}+u_k,\underline{t}}v=\sum_{\underline{j}\leqslant\underline{s}-\underline{i}} g_{\underline{j}}\left\{|G|^{-1}\sum_{\gamma\in\widehat{G}}\gamma\big((ab)^{\underline{i}+u_k}\big)c_{\underline{j}}(\gamma)\right\}x^{\underline{s}-\underline{i}-\underline{j}}\eta^{\underline{t}+u_k+\underline{i}+\underline{j}}\otimes 1_\lambda.$$

Applying the Gauss' binomial formula to $c_{\underline{j}}(\gamma)$ we have

$$\sum_{\gamma\in\widehat{G}}\gamma\big((ab)^{\underline{i}+u_k}\big)c_{\underline{j}}=\sum_{\underline{l}\leqslant\underline{j}}\kappa_{\underline{l}}\sigma_{\underline{l}},\quad \kappa_{\underline{l}}\in\Bbbk,\quad\text{where} \tag{$**$}$$

$$\sigma_{\underline{l}}=\sum_{\gamma\in\widehat{G}}\gamma\big((ab)^{\underline{i}+u_k+\underline{l}}\big) \tag{$***$}$$

Further we note that, as $\underline{i}+\underline{l}\leqslant\underline{s}$, if $s_k<m_k-1$, we have $\underline{0}<\underline{i}+u_k+\underline{l}<\underline{m}$, hence the sum $(***)$ is zero for all $\underline{l}$, and therefore the sum $(**)$ is zero for all $\underline{j}$. Same conclusion holds if $s_k=m_k-1$, but $j_k<s_k$. What remains to be considered is the case when $l_k=j_k=m_k-1-i_k$. Now $l_k+i_k+1=m_k$, hence the exponent on η_k in $\eta^{\underline{t}+u_k+\underline{i}+\underline{j}}$ is $\geqslant m_k$, and therefore this term vanishes. Thus the sum $(**)$ is always zero, and (2) is done.

(3) can be reduced to (1) by replacing $\underline{t}$ with $\underline{t}-u_k$. $\square$

From now on we assume that $\mathcal{D}$ is half-clean. We need one more commutation formula.

Lemma 4.18. *For every $s\geqslant 1$ and every $k\in\underline{n}$ there are functions r_i^s of q_k, $1\leqslant i\leqslant s$, such that*

$$\big[x_k^s,\xi_k\big]_{\phi^s_{x_k}}=(s)_{q_k}x_k^{s-1}\big(a_k-q_k^{-(s-1)}\chi_k\big)+\sum_{i=1}^{s-1}r_i^s x_k^{s-1-i}\eta_k^i\chi_k. \tag{4.37}$$

Proof. As before we drop the subscript k throughout. We induct on s noting that the formula holds for $s = 1$ by (4.22). To carry out the induction step we use Lemma 4.1(2), namely

$$\left[x^{s+1}, \xi\right]_{\phi^{s+1}} = x^s\left[x, a^{-s} \rightharpoonup \xi\right]_{\phi} + \left[x^s, \xi\right]_{\phi^s} x$$

(which by (4.13), (4.22), and the induction hypothesis equals)

$$x^s(a-\chi) + (s)_q x^{s-1}\left(a - q^{-(s-1)}\chi\right)x + \sum_{i=1}^{s-1} r_i^s x^{s-1-i}\eta^i \chi x.$$

For every $m = s - i$, $i > 0$, the coefficient of x^m is of the form $r_i^{s+1}\eta^{i+1}\chi$ by (4.34) as needed. It remains to compute the coefficient of x^s. Using (4.17), we have $\chi x = q^{-1}x\chi + (q^2-1)\eta\chi$, hence $(a - q^{(s-1)}\chi)x = x(qa - q^{-s}\chi) + \kappa\eta\chi$ for some $\kappa \in \Bbbk$. Thus this coefficient equals $(a-\chi) + (s)_q(qa - q^{-s}\chi) = (s+1)_q(a - q^{-s}\chi)$ and the proof is complete. $\square$

In what follows we set $v_{\underline{s},\underline{t}} = 0$ if s_k or t_k is zero for some $k \in \underline{n}$. We derive now action of η_k, ξ_k on $I(\lambda)$.

Proposition 4.19. *For each $k \in \underline{n}$ and $\underline{s}, \underline{t} \in \mathbb{Z}(\underline{m})$ there are roots of unity θ, θ' such that*

(1) $\eta_k v_{\underline{s},\underline{t}} = \theta v_{\underline{s},\underline{t}+u_k}$,
(2) $\xi_k v_{\underline{s},\underline{t}} = \theta'(s_k)_{q_k}(\lambda(a_k\chi_k^{-1}) - q_k^{-(s_k-1)})v_{\underline{s}-u_k,\underline{t}}$.

Proof. (1) We start off by deriving the action of η_k, ξ_k on idempotents $e_{\underline{s},\underline{t}}$. Pick $\mu \in \Gamma$. We have

$$\begin{aligned}\eta_k e_\mu &= |\Gamma|^{-1}\sum \mu\left((g\gamma)^{-1}\right)\eta_k g\gamma = |\Gamma|^{-1}\left(\sum \mu\left((g\gamma)^{-1}\right)\chi_k\left(g^{-1}\right)\widehat{b_k}\left(\gamma^{-1}\right)\right)\\ &= e_\nu \eta_k, \quad \text{where } \nu = \mu\widehat{b_k}\chi_k.\end{aligned}$$

Setting $\mu = \lambda_{\underline{s},\underline{t}}$ we obtain the first identity, namely

$$\eta_k e_{\underline{s},\underline{t}} = e_{\underline{s},\underline{t}+u_k}\eta_k. \tag{4.38}$$

For ξ_k we can show similarly that

$$\xi_k e_{\underline{s},\underline{t}} = e_{\underline{s}-u_k,\underline{t}}\xi_k. \tag{4.39}$$

We continue with part (1). In order to apply (4.38) we must expand $\eta_k x^{\underline{s}}\eta^{\underline{t}} \otimes 1_\lambda$ in the standard basis of $I(\lambda)$. By (4.27) η_k skew commute with x_l for all $l \neq k$. For $l = k$ we use Lemma 4.11 to pass η_k over $x_k^{s_k}$. Noting that η_k skew commutes with every η_l we arrive at the equality

$$\eta_k x^{\underline{s}}\eta^{\underline{t}} \otimes 1_\lambda = \theta x^{\underline{s}}\eta^{\underline{t}+u_k} \otimes 1_\lambda + \sum_{i=1}^{s} h_i x^{\underline{s}-iu_k}\eta^{\underline{t}+(i+1)u_k} \otimes 1_\lambda \tag{4.40}$$

where h_i are some elements of $\Bbbk$. By part (1) of the preceding proposition $e_{\underline{s},\underline{t}+u_k}$ annihilates $x^{\underline{s}-iu_k}\eta^{\underline{t}+(i+1)u_k} \otimes 1_\lambda$ for every $i \geqslant 1$, which gives (1).

(2) We begin by working out an expansion of $\xi_k x^{\underline{s}}\eta^{\underline{t}} \otimes 1_\lambda$ in the standard basis of $I(\lambda)$. Set $\underline{s}' = (s_1, \ldots, s_{k-1}, 0, \ldots, 0)$ and $\underline{s}'' = (0, \ldots, 0, s_{k+1}, \ldots, s_n)$. Noting that ξ_k commutes with x_l for every $l \neq k$

and commutes with all η_l, as well as the equality $\xi_k.1_\lambda = 0$, we have with the help of Lemma 4.18 the first equality, namely

$$\xi_k x^{\underline{s}} \eta^{\underline{t}} \otimes 1_\lambda = (s_k)_{q_k} w_0 + \sum_{i=1}^{s_k-1} \kappa_i w_i, \tag{$*$}$$

where $\kappa_i \in \mathbb{k}$, $w_0 = x^{\underline{s}'} x_k^{s_k-1}(a_k - q_k^{-(s_k-1)}\chi_k) x^{\underline{s}''} \eta^{\underline{t}} \otimes 1_\lambda$ and $w_i = x^{\underline{s}'} x_k^{s_k-i-1} \eta_k^i \chi_k x^{\underline{s}''} \eta^{\underline{t}} \otimes 1_\lambda$.

The rest of the proof will be carried out in steps. (i) By (3.3) and (4.17) $\chi_k x_l = q_{lk}^{-1} x_l \chi_k$, and also $a_k x_l = q_{lk}^{-1} x_l a_k$ for $l \neq k$. Further a_k skew commutes with $\eta^{\underline{t}}$ with scalar $\chi^{\underline{t}}(a_k)$ and χ_k skew commutes with $\eta^{\underline{t}}$ with scalar $\chi_k(b^{\underline{t}})$. As $\chi^{\underline{t}}(a_k) = \chi_k(b^{\underline{t}})$ by condition (D1), it follows that $w_0 = \theta'(\lambda(a_k\chi_k^{-1} - q^{-(s_k-1)}) x^{\underline{s}-u_k} \eta^{\underline{t}} \otimes 1_\lambda$.

(ii) We claim that $\eta_k^i \chi_k$ skew commutes with x_l for every $l \neq k$, or, equivalently, $\delta_{x_l}(\eta_k^i \chi_k) = 0$. Indeed, $\delta_{x_l}(\eta_k) = 0$ holds by (4.27) and $\delta_{x_l}(\chi_k) = 0$ follows from the first line of proof of (i). Since δ_{x_l} is a skew-derivation, the claim follows. As η_k, χ_k skew commute with all η_l it becomes clear that $w_i = \kappa x^{\underline{s}-(i+1)u_k} \eta^{\underline{t}+iu_k} \otimes 1_\lambda$ for some $\kappa \in \mathbb{k}$.

From (i) and (ii) we have the expansion

$$\xi_k x^{\underline{s}} \eta^{\underline{t}} \otimes 1_\lambda = \theta'(s_k)_{q_k} \lambda\big(a_k \chi_k^{-1} - q_k^{-(s_k-1)}\big) x^{\underline{s}-u_k} \eta^{\underline{t}} \otimes 1_\lambda + \sum_{i=1}^{s_k-1} \kappa_i x^{\underline{s}-(i+1)u_k} \eta^{\underline{t}+iu_k} \otimes 1_\lambda. \tag{4.41}$$

Applying $e_{\underline{s}-u_k,\underline{t}}$ to equality (4.41) we arrive by Proposition 4.17(1) at the desired result. $\square$

The next lemma gives a commutation relation between some generators of H and primitive idempotents.

Lemma 4.20. *For every $\in \underline{n}$ and $\underline{s}, \underline{t} \in \mathbb{Z}(\underline{m})$*

(1) $x_k e_{\underline{s},\underline{t}} = e_{\underline{s}+u_k,\underline{t}} x_k + q_k(e_{\underline{s},\underline{t}+u_k} - e_{\underline{s}+u_k,\underline{t}})\eta_k$,
(2) $y_k e_{\underline{s},\underline{t}} = e_{\underline{s},\underline{t}-u_k} y_k + q_k(e_{\underline{s},\underline{t}-u_k} - e_{\underline{s}-u_k,\underline{t}}) b_k \xi_k$.

Proof. Combining Lemmas 4.1, 4.4, 4.6 and Proposition 4.3 we compute

$$x_k g\gamma = \chi_k\big(g^{-1}\big)\gamma(a_k) g\gamma x_k - q_k \chi_k\big(g^{-1}\big)\big(\gamma(a_k) - \gamma\big(b_k^{-1}\big)\big) g\gamma \eta_k,$$
$$y_k g\gamma = \chi_k(g)\gamma(b_k) g\gamma y_k + q_k \chi_k(g)\big(\gamma(b_k) - \gamma\big(a_k^{-1}\big)\big) g\gamma b_k \xi_k.$$

It follows that for every $\mu \in \widehat{\Gamma}$ $x_k e_\mu$ splits up into the sum $x_k e_\mu = e_{\mu'} x_k - q_k(e_{\mu'} - e_{\mu''})\eta_k$ where $\mu' = \mu \chi_k \widehat{a_k^{-1}}$ and $\mu'' = \mu \chi_k \widehat{b_k}$. Similarly $y_k e_\mu = e_{\mu'} y_k + q_k(e_{\mu'} - e_{\mu''}) b_k \xi_k$ where $\mu' = \mu \chi_k^{-1} \widehat{b_k^{-1}}$ and $\mu'' = \mu \chi_k^{-1} \widehat{a_k}$. Setting $\mu = \lambda_{\underline{s},\underline{t}}$ we obtain the lemma. $\square$

We can now describe action of x_k on $I(\lambda)$. Briefly, x_k acts as a *raising* operator, however, not literally, because the set of weights is not ordered.

Proposition 4.21. *For every $k \in \underline{n}$ and $\underline{s}, \underline{t} \in \mathbb{Z}(\underline{m})$ there are roots of unity θ, θ' such that*

$$x_k v_{\underline{s},\underline{t}} = \theta v_{\underline{s}+u_k,\underline{t}} + \theta' v_{\underline{s},\underline{t}+u_k}.$$

Proof. Since x_k skew commutes with every x_l, $l \neq k$, $x_k(x^{\underline{s}}\eta^{\underline{t}}) \otimes 1_\lambda = \theta x^{\underline{s}+u_k}\eta^{\underline{t}} \otimes 1_\lambda$. By the previous lemma the latter monomial contributes $v_{\underline{s}+u_k,\underline{t}}$ to $x_k v_{\underline{s},\underline{t}}$. We move on to $w := \eta_k x^{\underline{s}}\eta^{\underline{t}} \otimes 1_\lambda$. From the expansion (4.40) and Proposition 4.17(1) we see that $e_{\underline{s},\underline{t}+u_k} w = \theta' v_{\underline{s},\underline{t}+u_k}$. On the other hand part (2) of that proposition gives $e_{\underline{s}+u_k,\underline{t}} w = 0$, and this completes the proof. □

We isolate one step of the next proposition in

Lemma 4.22. *For every $k \in \underline{n}$ and $t < m_k$ there holds*

$$y_k \eta_k^t = q_k^t \eta_k^t y_k - (t)_{q_k} \eta^{t-1}\left(q_k^t \chi_k b_k - \epsilon\right).$$

Proof. We use induction on t, the case $t = 1$ is covered by Lemma 4.8. Let $\phi : H^* \to H^*$ be the automorphism $\phi(\alpha) = b_k \rightharpoonup \alpha, \alpha \in H^*$. Below we drop the subscript k on y_k, η_k. For the induction step we make use of (1.14) which allows us to write

$$\begin{aligned} {}_\phi\left[y, \eta^{t+1}\right] &= {}_\phi\left[y, \eta^t\right]\eta + q^t \eta^t {}_\phi[y, \eta] \\ &= -(t)_q \eta^{t-1}\left(\left(q^t \chi b - \epsilon\right)\eta - q^t \eta^t (\chi b - \epsilon)\right), \end{aligned}$$

where for the last line we used the induction hypothesis and the basis of induction. One can check easily the equality $\chi b \eta = q^2 \eta \chi b$ which leads to the equality ${}_\phi[y, \eta^{t+1}] = -\eta^t(q^t((t)_q q + 1)\chi b - ((t)_q + q^t)\epsilon) = -\eta^t (t+1)_q (q^t \chi b - \epsilon)$. □

We finish this section by showing that y_k acts as a lowering operator. More precisely

Proposition 4.23. *For every $k \in \underline{n}$ and $\underline{s}, \underline{t} \in \mathbb{Z}(\underline{m})$ there are roots of unity θ, θ' such that*

$$\begin{aligned} y_k v_{\underline{s},\underline{t}} &= \theta (t_k)_{q_k}\left(\lambda(\chi_k b_k) - q_k^{-(t_k-1)}\right) v_{\underline{s},\underline{t}-u_k} \\ &\quad + \theta'(s_k)_{q_k}\left(\lambda\left(a_k \chi_k^{-1}\right) - q_k^{-(s_k-1)}\right) v_{\underline{s}-u_k,\underline{t}}. \end{aligned} \tag{4.42}$$

Proof. In view of Lemma 4.20 we want to express $y_k x^{\underline{s}}\eta^{\underline{t}} \otimes 1_\lambda$ and $\xi_k x^{\underline{s}}\eta^{\underline{t}} \otimes 1_\lambda$ in the standard basis of $I(\lambda)$. For the second of those vectors the expansion is given by (4.41). Multiplying (4.41) by $e_{\underline{s}-u_k,\underline{t}}$ and $e_{\underline{s},\underline{t}-u_k}$ in turn, we get the second summand of (4.42) and zero, respectively, by Proposition 4.17(1), (3).

We turn now to $w = y_k x^{\underline{s}}\eta^{\underline{t}} \otimes 1_\lambda$. y_k skew commutes with x_l and η_l for $l \neq k$. For $l = k$ the product $y_k x_k^{s_k}$ can be straightened out by Lemma 1.2(2). In addition $a_k b_k$ commutes with x_l for $l \neq k$ and skew commutes with all η_l. It follows that

$$w = \kappa x^{\underline{s}} y_k \eta^{\underline{t}} \otimes 1_\lambda + \mu x^{\underline{s}-u_k}\eta^{\underline{t}} \otimes 1_\lambda,$$

where $\kappa, \mu \in \Bbbk$ with κ a root of unity. Since $e_{\underline{s},\underline{t}-u_k}$ annihilates the second monomial in w by Proposition 4.17(3), we turn to the first summand in w. Using the preceding lemma and the fact that $y_k.1_\lambda = 0$ w reduces to the form $\theta(t_k)_{q_k}(\lambda(\chi_k b_k) - q_k^{-(t_k-1)}) x^{\underline{s}}\eta^{\underline{t}-u_k} \otimes 1_\lambda$ which, upon multiplication by $e_{\underline{s},\underline{t}-u_k}$, becomes the first summand of (4.42). □

4.3.2. The main theorems

In this section we will show that representation theory of $D(H)$ in $I(\lambda)$ follows the pattern established for H in Theorem 3.7.

Definition 4.24. For $\lambda \in \widehat{\Gamma}$ we let $S(\lambda)$ stand for all $j \in \underline{n}$ satisfying the condition

$$\lambda\big(a_j \chi_j^{-1}\big) = q_j^{-e_j}, \quad \text{or} \tag{4.43}$$

$$\lambda(b_j \chi_j) = q_j^{-e'_j}, \tag{4.44}$$

for some $0 \leqslant e_j, e'_j \leqslant m_j - 2$.

Those constants depend on λ. We write them as $e_j(\lambda), e'_j(\lambda)$ when this dependance must be emphasized.

For an n-tuple $\underline{a} \in \mathbb{Z}^n$ we define *the rank* of $\underline{a}$ as the number $\operatorname{rk}(\underline{a})$ of all $j \in S(\lambda)$ satisfying $a_j \geqslant e_j + 1$. We define *the rank* of $x^{\underline{s}}\eta^{\underline{t}} \otimes 1_\lambda$ by $\operatorname{rk}(x^{\underline{s}}\eta^{\underline{t}} \otimes 1_\lambda) = \operatorname{rk}(\underline{s}) + \operatorname{rk}(\underline{t})$.

Lemma 4.25. *A weight vector $v_{\underline{s},\underline{t}}$ is primitive if and only if $s_j = 0$, $e_j + 1$ or $t_j = 0$, $e'_j + 1$ for every $j \in S(\lambda)$, and $s_k = 0$, $t_k = 0$ for every $k \notin S(\lambda)$.*

Proof. This follows immediately from Propositions 4.19 and 4.23. $\square$

Let us denote the D-module generated by a set X by $\langle X\rangle$.

Proposition 4.26. *$R(\lambda)$ is generated by the primitive elements of rank* 1.

Proof. Suppose v is primitive. From definition of primitivity we have that $\langle v\rangle$ is the span of all $x^{\underline{s}}\eta^{\underline{t}}v$. Let now v has rank 1. Then v has the form $w_j = v_{(e_j+1)u_j,\underline{0}}$ or $w'_j = v_{\underline{0},(e'_j+1)u_j}$. From Propositions 4.19 and 4.21 we see that $\langle v\rangle$ is the span of either all $v_{\underline{c},\underline{d}}$ with $\underline{c} \geqslant (e_j+1)u_j$ and $\underline{d} \geqslant \underline{0}$ or $v_{\underline{c},\underline{d}}$ with $\underline{c} \geqslant \underline{0}$ and $\underline{d} \geqslant (e'_j+1)u_j$. Therefore were $R(\lambda) \neq \sum_{j\in S(\lambda)} \langle w_j\rangle + \langle w'_j\rangle$ there would be a weight vector $w \in R(\lambda)$ involving $v_{\underline{s},\underline{t}}$ with $s_j \leqslant e_j$ and $t_j \leqslant e'_j$ for all $j \in S(\lambda)$. We may assume that $v_{\underline{s},\underline{t}}$ is the minimal such in the sense that $R(\lambda)$ does not contain weight vectors whose expansion in the standard basis involves $v_{\underline{c},\underline{d}}$ with either $\underline{c} < \underline{s}$ and $\underline{d} \leqslant \underline{t}$ or $\underline{c} \leqslant \underline{s}$ and $\underline{d} < \underline{t}$. Let

$$w = v_{\underline{s},\underline{t}} + \sum c_{\underline{s}',\underline{t}'} v_{\underline{s}',\underline{t}'}, \quad c_{\underline{s}',\underline{t}'} \in \Bbbk,$$

be the expansion of w in the standard basis of $I(\lambda)$. If $\underline{s} = \underline{0} = \underline{t}$, then $1 \otimes 1_\lambda + \sum_{(\underline{s},\underline{t})\neq(\underline{0},\underline{0})} v_{\underline{s},\underline{t}} \in I(\lambda)$, hence, as x_i, η_i generate a nilpotent subalgebra, $1 \otimes 1_\lambda \in R(\lambda)$, a contradiction. Else, set $w' = \xi_k w$ and $w'' = y_k w$. Then, either $s_k \neq 0$ for some k, hence w' involves a weight vector smaller than $v_{\underline{s},\underline{t}}$ by Proposition 4.19, or $\underline{s} = \underline{0}$, but $t_k \neq 0$, and then the same holds for w'' by Proposition 4.23. In either case we arrive at a contradiction. $\square$

For the dimension formula we introduce some subsets of $S(\lambda)$. We let $S^{(1)}(\lambda), S^{(2)}(\lambda)$ and $S^{(3)}(\lambda)$ be defined by the conditions

$$S^{(1)}(\lambda) = \big\{ j \in S(\lambda) \bigm| j \text{ satisfies only (4.43)}\big\},$$
$$S^{(2)}(\lambda) = \big\{ j \in S(\lambda) \bigm| j \text{ satisfies only (4.44)}\big\},$$
$$S^{(3)}(\lambda) = S^{(1)}(\lambda) \cap S^{(2)}(\lambda).$$

Corollary 4.27. *For every $\lambda \in \widehat{\Gamma}$*

$$\dim L(\lambda) = \prod_{j\notin S(\lambda)} m_j^2 \prod_{j\in S^{(1)}(\lambda)} (e_j+1)m_j \prod_{j\in S^{(2)}(\lambda)} \big(e'_j+1\big)m_j \prod_{j\in S^{(3)}(\lambda)} (e_j+1)\big(e'_j+1\big).$$

Proof. By definition $L(\lambda) = I(\lambda)/R(\lambda)$. By the above proposition $L(\lambda)$ is the span of all $v_{\underline{s},\underline{t}}$ with $s_j \leqslant e_j$ and $t_j \leqslant e'_j$ for $j \in S^{(1)}(\lambda)$ or $j \in S^{(2)}(\lambda)$, respectively, and arbitrary integers within $[0, m_j - 1]$, otherwise. $\square$

We refer to Section 3.2 for a discussion of the Loewy and socle series. We denote by $\ell(\lambda)$ the Loewy length of $I(\lambda)$.

Theorem 4.28. (1) *For every $m < \ell(\lambda)$, $R^m(\lambda)$ is generated by the primitive vectors of rank m.*

(2) *The radical and the socle series coincide.*

(3) *The Loewy layers $\mathcal{L}^m := R^m(\lambda)/R^{m+1}(\lambda)$ are given by the formula*

$$\mathcal{L}^m \simeq \bigoplus \bigl\{ L(\mu) \bigm| \mu \text{ is weight of primitive basis vector of rank } m \bigr\}.$$

(4) $\ell(\lambda) = |S^{(1)}(\lambda)| + |S^{(2)}(\lambda)| + 1$.

Proof. (1)–(4). We abbreviate $R(\lambda)$ to R. We induct on m. The fact that $I(\lambda)$ is generated by $1 \otimes 1_\lambda$ gives the basis of induction. Suppose (1) holds for R^m. The induction step will be carried out in steps.

(i) Let $v_{\underline{s},\underline{t}}$ be a primitive vector of rank m, and $\mu = \lambda_{\underline{s},\underline{t}}$ be its weight. We claim that $S^{(i)}(\mu) = S^{(i)}(\lambda)$ for $i = 1, 2, 3$. Namely, $e_j(\mu) = e_j(\lambda)$ if $s_j = 0$, and $e_j(\mu) = m_j - e_j(\lambda) - 2$, otherwise. Similarly, $e'_j(\mu) = e'_j(\lambda)$ if $t_j = 0$, and $e'_j(\mu) = m_j - e'_j(\lambda) - 2$, otherwise. This follows from the calculation

$$\begin{aligned}
\chi^{\underline{s}+\underline{t}}\widehat{a^{-\underline{s}}b^{\underline{t}}}\bigl(a_k \chi_k^{-1}\bigr) &= \chi^{\underline{s}+\underline{t}}(a_k)\chi_k\bigl(a^{\underline{s}}b^{-\underline{t}}\bigr) \\
&= \Bigl[\prod_{m \neq k} \chi_m^{s_m}(a_k)\chi_m^{t_m}(a_k)\Bigr]\chi_k^{s_k}(a_k)\chi_k^{t_k}(a_k)\Bigl[\prod_{m \neq k} \chi_k\bigl(a_m^{s_m}\bigr)\chi_k\bigl(b^{-t_m}\bigr)\Bigr],
\end{aligned}$$

$$\chi_k(a_k^{s_k})\chi_k(b_k^{-t_k}) = \prod_{m \neq k}\bigl[\chi_m(a_m)\chi_k(a_m)\bigr]^{s_m}\bigl[\chi_m(a_k)\chi_k\bigl(b_m^{-1}\bigr)\bigr]^{t_m},$$

$$\chi_k^{2s_k}(a_k) = \chi_k^{2s_k}(a_k)$$

with the last equality by the datum conditions (D1)–(D2). Similarly $\chi^{\underline{s}+\underline{t}}\widehat{a^{-\underline{s}}b^{\underline{t}}}(b_k\chi_k) = \chi_k^{2t_k}(b_k)$. Therefore $\mu(a_k\chi_k^{-1}) = \lambda(a_k\chi_k^{-1})\chi_k^{2s_k}(a_k)$ and $\mu(b_k\chi_k) = \lambda(b_k\chi_k)\chi_k^{2t_k}(b_k)$. It follows that if $s_k = 0$ or $t_k = 0$, then the value of μ and λ at $a_k\chi_k^{-1}$ or $b_k\chi_k$ coincide. Else, $s_k = e_k(\lambda) + 1$ or $t_k = e'_k(\lambda) + 1$, whence $\mu(a_k\chi_k^{-1}) = \lambda(a_k\chi_k^{-1})q_k^{2(e_k(\lambda)+1)} = q_k^{-(m_k - e_k(\lambda) - 2)}$, or similarly $\mu(b_k\chi_k) = q_k^{-(m_k - e'_k(\lambda) - 2)}$. This proves our claim.

(ii) Since $v_{\underline{s},\underline{t}}$ is primitive of weight μ there is a D-epimorphism $\phi : I(\mu) \to Dv_{\underline{s},\underline{t}}$ determined on the generator by $\phi : 1 \otimes 1_\mu \mapsto v_{\underline{s},\underline{t}}$. It follows that $R(Dv_{\underline{s},\underline{t}}) = \phi(R(\mu))$, hence, by the preceding proposition, we have that $R(Dv_{\underline{s},\underline{t}}) = \sum D\phi(w)$ where w runs over the primitive vectors of $I(\mu)$ of rank 1. Put $f_j = e_j(\mu) + 1$ and $f'_j = e'_j(\mu) + 1$ for brevity. We know that w is either $w_j = e_\nu(x_j^{f_j} \otimes 1_\mu)$ or $w'_j = e_{\nu'}(\eta_j^{f'_j} \otimes 1_\mu)$ for some $j \in S(\mu)$ where $\nu = \mu_{f_j u_j, \underline{0}}$ and $\nu' = \mu_{\underline{0}, f'_j u_j}$ are weights of w_j and w'_j, respectively. We consider two cases.

(a) Suppose $w = w_j$. Since $\mu = \lambda_{\underline{s},\underline{t}}$ we have

$$\nu = \lambda_{\underline{s},\underline{t}}\widehat{a^{-f_j u_j}}\chi^{f_j u_j} = \lambda_{\underline{s}+f_j u_j, \underline{t}}.$$

Therefore $\phi(w) = e_\nu x_j^{f_j} v_{\underline{s},\underline{t}}$ which by Lemma 4.21 is seen to be

$$e_\nu\Bigl(\sum_{l+k=f_j} c_{l,k} v_{\underline{s}+l u_j, \underline{t}+k u_j}\Bigr) \quad \text{for some } c_{l,k} \in \Bbbk.$$

Furthermore, the equality of characters $\lambda_{\underline{s}+lu_j,\underline{t}+ku_j}=\nu$ is equivalent to $a_j^{-l}b_j^k=a_j^{-f_j}$ which, in view of $f_j=l+k$, reduces to $(a_jb_j)^k=1$. By our assumption on datum the last condition holds only for $k=0$. Thus $e_\nu x_j^{f_j}v_{\underline{s},\underline{t}}=cv_{\underline{s}+f_ju_j,\underline{t}}$, $c\in\mathbb{k}$, and $c\neq 0$ by another application of Proposition 4.21. The last formula and part (i) make it clear that if $j\in S^{(1)}(\mu)=S^{(1)}(\lambda)$ and $s_j=0$, then $\phi(w)$ is a primitive vector of rank $m+1$. If $s_j=e_j(\lambda)+1$, then $f_j=e_j(\mu)+1=m_j-e_j(\lambda)-1$ again by part (i), hence $f_j+s_j=m_j$, whence $\phi(w)=0$.

(b) We let $w=w'_j$. Proposition 4.3(1) and Lemma 4.4 show that $\eta_j^{f'_j}\otimes 1_\mu$ has weight ν', hence $w'_j=\eta_j^{f'_j}\otimes 1_\mu$. By Proposition 4.19(1) $\phi(w)=cv_{\underline{s},\underline{t}+f'_ju_j}, c\neq 0$. If $j\in S^{(2)}(\lambda)$ and $t_j=0$, then $f'_j=e'_j(\lambda)+1$, hence $\phi(w)$ is a primitive vector of rank $m+1$. Otherwise, by an argument as above, $f'_j+t_j=m_j$, which gives $\phi(w)=0$. It remains to note that the primitive vectors constructed in (a) and (b) account for all primitive vectors of rank $m+1$.

For (4) we note that every element of $S\setminus S^{(3)}$ contributes a one and every element of $S^{(3)}$ contributes a two to the rank function. Thus the largest value the rank function can have is $|S\setminus S^{(3)}|+2|S^{(2)}|=|S^{(1)}|+|S^{(2)}|$.

(2) We begin the argument as in the same part of Theorem 3.7. We assume by the reverse induction on m that $R^{m+1}=\Sigma_{\ell-m-1}$ and we let $M=I(\lambda)/R^{m+1}$. We want to prove the equality $R^m/R^{m+1}=\Sigma(M)$. Assuming it does not hold, there is a simple D-module L in $\Sigma(M)$ not contained in R^m/R^{m+1}. Let k be the largest integer such that $L\subset R^k/R^{m+1}$. We denote by $\overline{v}$ the image of $v\in I(\lambda)$ in M. We define $\operatorname{rk}(\overline{v_{\underline{s},\underline{t}}})=\operatorname{rk}(v_{\underline{s},\underline{t}})$.

Our first step is to show that M is the free span of all $\overline{v_{\underline{s},\underline{t}}}$ with $\operatorname{rk}(v_{\underline{s},\underline{t}})<m$. Indeed, a $v_{\underline{s},\underline{t}}$ of rank $\geqslant m+1$ is characterized by the property that $s_j\geqslant e_j+1$ or $t_{j'}\geqslant e'_{j'}+1$ for some $j,j'\in S(\lambda)$ of the total number greater than m. Therefore there is a primitive $v_{\underline{c},\underline{d}}$ such that $s_j=c_j+k_j$ and $t_{j'}=d_{j'}+l_{j'}$ for some positive integers k_j and $l_{j'}$. Now a glance at Propositions 4.19 and 4.21 leads to conclusion that $v_{\underline{s},\underline{t}}$ lies in $x^{\underline{i}}\eta^{\underline{j}}v_{\underline{c},\underline{d}}$. Since by part (1) R^{m+1} is generated by primitive vectors of rank $m+1$, the assertion follows.

Let u be a generator of L written in basis B as

$$u=\sum c_{\underline{s},\underline{t}}\overline{v_{\underline{s},\underline{t}}},\quad 0\neq c_{\underline{s},\underline{t}}\in\mathbb{k}. \tag{$*$}$$

The sum u_k of all $\overline{v_{\underline{c},\underline{d}}}$ of rank k occurring in $(*)$ is nonzero. Let us call the number of terms in the sum $(*)$ for u_k *the length* of u_k. We pick a generator u with u_k of the smallest length. Fix one $\overline{v_{\underline{c},\underline{d}}}$ involved in u_k. Assuming $k<m$ we can find $j\in S(\lambda)$ such that $c_j<e_j(\lambda)+1$ or $d_j<e'_j(\lambda)+1$. Suppose $c_j<e_j(\lambda)+1$. Set $l_j=e_j(\lambda)+1-c_j$ and $\nu=\lambda_{\underline{c}+l_ju_j,\underline{d}}$. By the argument used in the proof of (1, (iia)) for every vector $v_{\underline{s},\underline{t}}$, $e_\nu x_j^{l_j}v_{\underline{s},\underline{t}}=\kappa v_{\underline{s}+l_ju_j,\underline{t}}$, $\kappa\in\mathbb{k}^\bullet$. Since $v_{\underline{c}+l_ju_j,\underline{t}}$ has rank $k+1$, we see that $e_\nu x_j^{l_j}u$ is a nonzero generator with a lesser number of basis monomials of rank k, a contradiction.

Assuming $d_j,e'_j(\lambda)+1$, set $l'_j{}'=e'_j(\lambda)+1-d_j$. Using the argument of part (1, (iib)) we deduce that $\eta^{l'_j{}'}u$ is a nonzero element of L with the lesser number of basis monomials of rank k. This completes the proof of (2).

(3) We keep notation of part (2). By part (1)

$$\mathcal{L}^m=\sum D.\overline{v_{\underline{s},\underline{t}}}, \tag{$**$}$$

where $v_{\underline{s},\underline{t}}$ runs over all primitive basis vectors of rank m. Fix one $\overline{v_{\underline{s},\underline{t}}}$. By Propositions 4.19 and 4.21 $D.\overline{v_{\underline{s},\underline{t}}}$ is the span of the set $B_{\underline{s},\underline{t}}=\{\overline{v_{\underline{c},\underline{d}}}\mid \operatorname{rk}(v_{\underline{c},\underline{d}})=m \text{ and } \underline{c}\geqslant\underline{s}, \underline{d}\geqslant\underline{t}\}$. The condition $\operatorname{rk}(v_{\underline{c},\underline{d}})=m$ forces $0\leqslant c_j\leqslant e_j$ if $j\in S^{(1)}(\lambda)$ and $s_j=0$, and, likewise, $0\leqslant d_j\leqslant e'_j$ if $j\in S^{(2)}(\lambda)$ with $t_j=0$. If $j\in S^{(1)}(\lambda)$ or $j\in S^{(2)}(\lambda)$ with $s_j=e_j+1$ or $t_j=e'_j+1$, then $0\leqslant c_j\leqslant m_j-e_j-2$ or $0\leqslant d_j\leqslant m_j-e'_j-2$, respectively. For $j\notin S(\lambda)$, c_j,d_j take on every value in $[m_j]$. Therefore by part (1, (i)) and the dimension formula of Corollary 4.27 $|B_{\underline{s},\underline{t}}|=\dim L(\lambda_{\underline{s},\underline{t}})$. As $D.\overline{v_{\underline{s},\underline{t}}}\supset L(\lambda_{\underline{s},\underline{t}})$ we obtain the equality

$D.\overline{v_{\underline{s},\underline{t}}} = L(\lambda_{\underline{s},\underline{t}})$. Now, were sum $(**)$ not direct, some $\overline{v_{\underline{s},\underline{t}}}$ would be a linear combination of elements of other $B_{\underline{s}',\underline{t}'}$. Since $\overline{v_{\underline{s},\underline{t}}}$ is a basis element, it would lie in some $D.\overline{v_{\underline{s}',\underline{t}'}}$. However, from Theorem 4.15 one sees that every simple D-module has a unique line of primitive elements, a contradiction. $\square$

The (neo)classical quantum groups of Drinfel'd, Jimbo and Lusztig have the group of grouplikes equal to the direct sum of cyclic subgroups generated by the grouplikes associated to the 'positive' or 'negative' half of skew-primitive generators. In the finite-dimensional case (see e.g. [21]) the orders of all q_i are odd. In general those two conditions on datum are independent of each other. We call datum $\mathcal{D}$ *classical* if either all $|q_i|$ are odd, or the elements $\{a_i, b_i\}_{i\in\underline{n}}$ are independent in the sense that they generate the subgroup equal to the direct sum of cyclic subgroups generated by a_i and b_i. We turn to liftings H with classical data. We will give a complete description of the lattice of submodules of $I(\lambda)$ for every $\lambda \in \widehat{\Gamma}$. This is possible because the lattice of D-submodules turns out to be distributive, a consequence of the next lemma.

Lemma 4.29. *Suppose $\mathcal{D}$ is classical. Then $\mathcal{D}$ is half-clean and the weights $\lambda_{\underline{s},\underline{t}}$, $\underline{s}, \underline{t} \in \mathbb{Z}(\underline{m})$ are distinct.*

Proof. The first assertion holds by definition if all $|q_i|$ are odd. Else, suppose the set $\{a_i, b_i\}_{i\in\underline{n}}$ is independent. Then $\prod_{i=1}^n (a_i b_i)^{t}_i = 1$ implies $a_i^{t_i} = 1$ for all i. Since $\chi_i(a_i) = q_i$ and q_i has order m_i, s is divisible by m_i, which proves that $\mathcal{D}$ is half-clean.

Moving on to the second claim we must show that $\underline{s} = \underline{s}'$ and $\underline{t} = \underline{t}'$ whenever $\widehat{a^{-\underline{s}} b^{\underline{t}}} \chi^{\underline{s}+\underline{t}} = \widehat{a^{-\underline{s}'} b^{\underline{t}'}} \chi^{\underline{s}'+\underline{t}'}$. This equation is equivalent to

$$a^{-\underline{s}} b^{\underline{t}} = a^{-\underline{s}'} b^{\underline{t}'},$$

$$\chi^{\underline{s}+\underline{t}} = \chi^{\underline{s}'+\underline{t}'}.$$

Set $p_i = s_i - s_i'$ and $r_i = t_i - t_i'$ for all i. We rewrite the above two equations as

$$a^{-\underline{p}} b^{\underline{r}} = 1. \tag{$*$}$$

$$\chi^{\underline{p}+\underline{r}} = 1. \tag{$**$}$$

Suppose $\{a_i, b_i\}_{i\in\underline{n}}$ are independent. Eq. $(*)$ implies equalities $a_i^{p_i} = 1$ and $b_i^{r_i} = 1$ for all i. As $\chi_i(a_i) = \chi_i(b_i) = q_i$ and the latter has order m_i, we see that p_i and r_i are divisible by m_i. Since $-m_i < p_i$, $r_i < m_i$ we conclude that $p_i = 0 = r_i$, and this holds for all i.

Next assume that all m_i are odd. We induct on n assuming the lemma holds for every datum on $< n$ points. Since for $\underline{n} = \emptyset$ the claim is vacuously true we proceed to the induction step. Applying χ_1 to the equality $(*)$ gives

$$\chi_1(a_1)^{-p_1} \chi_1(b_1)^{r_1} \prod_{i=2}^n \chi_1(a_i)^{-p_i} \chi_1(b_i)^{r_i} = 1. \tag{!}$$

Using datum conditions (D1)–(D2) we have $\chi_1(a_1) = \chi_1(b_1) = q_1$, $\chi_1(a_i)^{-1} = \chi_i(a_1)$ and $\chi_1(b_i) = \chi_i(a_1)$. Therefore equality $(!)$ takes on the form

$$q_1^{r_1 - p_1} \prod_{i=2}^n \chi_i(a_1)^{p_i + r_i} = 1.$$

Further, evaluating the left side of $(**)$ at a_1 we get the equality

$$q_1^{p_1 + r_1} \prod_{i=2}^n \chi_i(a_1)^{p_i + r_i} = 1.$$

It follows that $r_1 - p_1 \equiv r_1 + p_1 \bmod m_i$. In addition taking the value of the left side of $(**)$ at b_1 we have

$$q_1^{p_1+r_1} \prod_{i=2}^{n} \chi_i(b_1)^{p_i+r_i} = q_1^{p_1+r_1} \prod_{i=2}^{n} \chi_i(a_1)^{-(p_i+r_i)} = 1.$$

Comparing the last two equalities we see that $q_1^{2(p_1+r_1)} = 1$ whence $p_1 + r_1 \equiv 0 \bmod m_i$, as m_i is odd. It follows that both p_i and r_i are divisible by m_i, hence $p_1 = 0 = r_1$, and the proof is complete. $\square$

The above lemma makes it clear that all vectors $v_{\underline{s},\underline{t}}$, $\underline{s}, \underline{t} \in \mathbb{Z}(\underline{m})$ have distinct weights. Therefore every weight subspace $I_\mu(\lambda) := e_\mu I(\lambda)$ is one-dimensional. Since $I(\lambda)$ is a semisimple $\Bbbk\widehat{\Gamma}$-module $I(\lambda) \simeq \bigoplus_{\mu\in\widehat{\Gamma}} m_\mu L(\mu)$ where m_μ is the multiplicity of $L(\mu)$ in $I(\lambda)$. As $L(\mu)$ contains a μ-weight vector, $m_\mu \leqslant 1$ for all μ. Thus $I(\lambda)$ is a multiplicity free module for all $\lambda \in \widehat{\Gamma}$. We digress briefly into a general theory of such modules. For an alternate treatment see [1].

Let A be an algebra and M a left A-module of finite length with every simple A-module occurring at most once in a composition series of M. Let Λ be the submodule lattice of M. Λ is distributive by a standard criterion [8, II.13]. An element $J \neq 0$ will be called *local* (or join-irreducible [8]) if $A \subsetneqq J$ and $B \subsetneqq J$ imply $A + B \subsetneqq J$. Clearly the radical $R(J)$ is a unique maximal submodule of J. We let $R(J) = 0$ if J is simple. Let $\mathcal{J} = \mathcal{J}(M)$ denote the poset (partially ordered set) of local submodules ordered by inclusion. By [8, Cor. III.3] $\mathcal{J}$ forms a distinguished basis for Λ in the sense that every $X \in \Lambda$ has a unique representation as the sum of an irredundant set of local submodules. Let $\mathcal{S}$ be the set of all composition factors of M. It turns out that $\mathcal{J}$ also determines $\mathcal{S}$ and the way composition factors are 'stuck' together. The precise statement is:

Proposition 4.30. *In the foregoing notation*

(1) *Let $N = J_1 + \cdots + J_k$ be an irredundant sum of local submodules of M. Then each J_i is a maximal submodule of N,*

$$R(N) = \sum_{i=1}^{k} R(J_i) \quad \textit{and} \quad N/R(N) = \bigoplus_{i=1}^{k} J_i/R(J_i).$$

(2) *The mapping $J \mapsto J/R(J)$ sets up a bijection between $\mathcal{J}$ and $\mathcal{S}$.*
(3) *The composition length of M equals $|\mathcal{J}|$.*
(4) *For $L \in \mathcal{S}$ let $J(L)$ be the preimage of L under the map in* (2)*. For any two simple modules L and L', L occurs before L' in a composition series of M if and only if $J(L)$ and $J(L')$ are either incomparable in $\mathcal{J}$ or $J(L) \supset J(L')$.*

Proof. (1) Suppose N is as in (1). Let $K = \sum_{i=1}^{k} R(J_i)$. Then $J_i \cap K = R(J_i) + \sum_{k\neq i} J_i \cap R(J_k) = R(J_i)$ because $J_i \cap R(J_k) \subset J_i \cap J_k \subset R(J_i)$. Therefore $N/K \simeq \bigoplus_{i=1}^{k} J_i/R(J_i)$, hence $K \supset R(N)$. On the other hand, $J_i + R(N)/R(N)$ is semisimple for all i, hence $R(N) \cap J_i \subset R(J_i)$, whence $R(N) \subset K$.

(2) Taking filtration $M \supset R(M) \supset R^2(M) \supset \cdots \supset 0$ we see by part (1) that every composition factor of M is of the form $J/R(J)$ for some $J \in \mathcal{J}$. Thus the mapping $\mathcal{J} \to \mathcal{S}$, $J \mapsto J/R(J)$ is onto. Were $J/R(J) \simeq J'/R(J')$ for some $J \neq J'$, the multiplicity m of $J/R(J)$ in M would be $\geqslant 2$. For, if $J \supset J'$, then refining $J \supset J' \supset 0$ we get $m \geqslant 2$. Else, we refine $J + J' \supset J \supset 0$, and get $m \geqslant 2$, again. By our assumption that M is multiplicity free, the assertion follows.

(3) is [8, Lemma III.2]. It also follows immediately from part (2) as the composition length is $|\mathcal{S}| = |\mathcal{J}|$.

(4) Suppose there is a composition series of M with L preceding L'. Say $L = A/B$ and $L' = B/C$. By part (1) $J(L) \subset A$ and $J(L') \subset B$. Since $J(L) \not\subset B$, lest we have the multiplicity of $L \geqslant 2$, we see that

$J(L) \not\subset J(L')$. Suppose $J(L)$ and $J(L')$ are incomparable. The refining $M \supset J(L) + J(L') \supset J(L') \supset 0$ we get L before L', and refining $M \supset J(L) + J(l') \supset J(L) \supset 0$ reverses the order of their appearance. $\square$

We return to modules $I(\lambda)$ assuming the datum to be classical. For every primitive vector $v_{\underline{s},\underline{t}}$ we define pair of sets $(S'(v_{\underline{s},\underline{t}}), S''(v_{\underline{s},\underline{t}}))$ by $S'(v_{\underline{s},\underline{t}}) = \{j \in S(\lambda) \mid s_j = e_j + 1\}$ and $S''(v_{\underline{s},\underline{t}}) = \{j \in S(\lambda) \mid t_j = e'_j + 1\}$. We let $P = P(\lambda)$ denote the set of all pairs (S', S'') with $S' \subset S'(\lambda)$ and $S'' \subset S''(\lambda)$. We turn P into a poset by defining an ordering $(S', S'') \succcurlyeq (T', T'')$ if and only if $S' \subseteq T'$ and $S'' \subseteq T''$. The main result is

Theorem 4.31. *In the foregoing notations with $H = H(\mathcal{D})$ and $D = D(H)$ there holds*

(1) *A submodule J of $I(\lambda)$ is local if and only if J is generated by a primitive weight vector.*
(2) *The poset $\mathcal{J}$ of local submodules of $I(\lambda)$ is isomorphic to P.*
(3) *The composition length of $I(\lambda)$ equals $2^{\ell(\lambda)-1}$.*

Proof. (1) Since all $v_{\underline{s},\underline{t}}$ have distinct weights every submodule M of $I(\lambda)$ is the span of $M \cap B$ where $B = \{v_{\underline{s},\underline{t}} \mid \underline{s}, \underline{t} \in \mathbb{Z}(\underline{m})\}$. Therefore, if J is generated by a single basis vector $v \in B$, then were $J = K + M$, v would lie in K or N, hence J is local. Conversely, if J is local, then, as $J = \sum_{v \in J \cap B} Dv$, we have $J = Dv$ for some v.

Let $v = v_{\underline{s},\underline{t}}$ be a generator of J. If $s_k \neq 0$ for some $k \notin S(\lambda)$ or $s_k \neq e_k + 1$ for some $k \in S(\lambda)$, then by Propositions 4.19 and 4.21 $\xi_k v_{\underline{s},\underline{t}}$ also generates J. Thus we can assume that $s_k = 0$ for every $k \notin S(\lambda)$ and $s_k = 0, e_k + 1$ for every $k \in S(\lambda)$. We come to a similar conclusion about every t_k, viz. $t_k = 0$ if $k \notin S(\lambda)$ and $t_k = 0, e'_k + 1$ for every $k \in S(\lambda)$ by using Propositions 4.19 and 4.23. This proves (1).

(2) Let $\psi : \mathcal{J} \to P$ be the mapping sending J to $\psi(v) := (S'(v), S''(v))$ where v is a primitive generator of J. By primitivity of v, J is the span of all $x^{\underline{i}} \eta^{\underline{j}} v$. Therefore Propositions 4.19(2) and 4.21 give the relation $\psi(w) \preccurlyeq \psi(v)$ for every primitive weight vector $w \in J$. This shows first that if v and w generate J then $\psi(w) = \psi(v)$, so that ψ is well defined, and second that ψ is isotonic, which yields (2).

(3) By part (3) of the preceding proposition the composition length of $I(\lambda)$ equals $|\mathcal{J}|$, hence $|P|$. The latter is $2^{|S^{(1)}| + |S^{(2)}|}$ which yields the claim by Theorem 4.28(4). $\square$

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